Optimal control problems of fully coupled FBSDEs and viscosity solutions of Hamilton-Jacobi-Bellman equations

Juan Li
School of Mathematics and Statistics, Shandong University, Weihai, Weihai 264209, P. R. China.
E-mail: juanli@sdu.edu.cn
Qingmeng Wei*
School of Mathematics, Shandong University, Jinan 250100, P. R. China.
E-mail: weiqingmeng0207@163.com

November 11, 2012

Abstract.
In this paper we study stochastic optimal control problems of fully coupled forward-backward stochastic differential equations (FBSDEs). The recursive cost functionals are defined by controlled fully coupled FBSDEs. We study two cases of diffusion coefficients σ of FSDEs. We use a new method to prove that the value functions are deterministic, satisfy the dynamic programming principle (DPP), and are viscosity solutions to the associated generalized Hamilton-Jacobi-Bellman (HJB) equations. The associated generalized HJB equations are related with algebraic equations when σ depends on the second component of the solution (Y, Z) of the BSDE and doesn't depend on the control. For this we adopt Peng's BSDE method, and so in particular, the notion of stochastic backward semigroup in [16]. We emphasize that the fact that σ also depends on Z makes the stochastic control much more complicate and has as consequence that the associated HJB equation is combined with an algebraic equation, which is inspired by Wu and Yu [19]. We use the continuation method combined with the fixed point theorem to prove that the algebraic equation has a unique solution, and moreover, we also give the representation for this solution. On the other hand, we prove some new basic estimates for fully coupled FBSDEs under the monotonic assumptions. In particular, we prove under the Lipschitz and linear growth conditions that fully coupled FBSDEs have a unique solution on the small time interval, if the Lipschitz constant of σ with respect to z is sufficiently small. We also establish a generalized comparison theorem for such fully coupled FBSDEs.

Keywords. Fully coupled FBSDEs; value functions; stochastic backward semigroup; dynamic programming principle; viscosity solution

*Corresponding author.

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs) driven by a Brownian motion were first introduced by Pardoux and Peng [11] in 1990. They got the uniqueness and the existence theorem for nonlinear BSDEs under Lipschitz condition. The theory of BSDEs has been studied since then by many authors and has found various applications, namely in stochastic control (see Peng [13]), finance (see El Karoui, Peng and Quenez [6]), and partial differential equations (PDE) theory (see Peng [14], etc).

Related with the BSDE theory, the theory of fully coupled forward-backward stochastic differential equation (FBSDE) has been developing very dynamically. We will usually meet fully coupled FBSDEs which are used to describe state processes and the related cost functionals when some optimization problems are studied (see Cvitanić and Ma [4], Ma and Yong [10]). There are many results on the existence and the uniqueness of solutions of fully coupled FBSDEs. Antonelli [1] first studied fully coupled FBSDEs driven by Brownian motion on a "small" time interval with the fixed point theorem. As we know, there are mainly three methods to study fully coupled FBSDEs on an arbitrarily given time interval. The first one is a kind of "four-step scheme" approach (see Ma, Protter and Yong [8]) which combines PDE methods and methods of probability. With these methods, the authors of [8] proved the existence and the uniqueness for fully coupled FBSDEs on an arbitrarily given time interval, but they required the equations to be non-degenerate, i.e., the diffusion coefficients are non-degenerate. However, the PDE approach can not be used to deal with the case, when the coefficients are random. The second method is that of continuation which is purely probabilistic, see Hu and Peng [7], Pardoux and Tang [12], Peng and Wu [17], Yong [21]. They used the "monotonicity" condition on the coefficients which relaxes the above assumptions. The third method is motivated by the numerical approaches for some linear FBSDEs (see Delarue [5] and Zhang [23]). Using a probabilistic method, Wu [18] proved a comparison theorem for FBSDEs. It is a useful tool to study fully coupled FBSDEs. Recently, Ma, Wu, Zhang and Zhang [9] used a unified method to study fully coupled FBSDEs. For more details on fully coupled FBSDEs, the reader is referred to the book of Ma and Yong [10]; more recent works on FBSDEs refer to Yong [22], or Ma, Wu, Zhang and Zhang [9], and the references therein.

BSDE methods (for instance, a generalized dynamic programming principle (DPP) or the maximum principle for control problem), originally developed by Peng [13], [15], [16], for the stochastic control theory. Pardoux and Tang [12] associated fully coupled FBSDEs (without controls) with quasilinear parabolic PDEs, and proved the existence of viscosity solutions. The diffusion coefficient σ of the forward equation in the FBSDE they considered is supposed not to depend on Z. In Wu and Yu [19], [20], they proved the existence of a viscosity solution for quasilinear PDEs with the help of fully coupled FBSDE when σ depends on z, and the stochastic systems without controls. Inspired by above works we want to study the optimal control problems of fully coupled FBSDEs. If one considers controlled fully coupled FBSDEs the problem becomes more difficult, as well as when one tries to give a probabilistic representation for systems of generalized fully nonlinear Hamilton-Jacobi-Bellman (HJB) equations. Using a method of Buckdahn and Li [2], without assuming the coefficients to be Hölder-continuous with respect to the control variable, we prove that the value function is deterministic (Proposition 3.1) and it satisfies DPP (Theorem 3.1). Furthermore, we obtain the existence of viscosity solutions of the associated HJB equations (Theorem 4.1 and Theorem 4.2). For this we adopt Peng's BSDE method (see [16], or [2]). However, unlike the existing literature, in the present work the stochastic backward semigroup is defined through a fully coupled FBSDE. This makes techniques which are implied by the work with the stochastic backward semigroup much more subtle. This arises, in particular, in the proofs of the Theorems 4.1 and 4.2 showing that the value function is a viscosity solution: a "classical" approach would lead to BSDEs with quadratic growth in (y, z). To avoid this, a deeper study of properties of fully coupled FBSDEs on short time intervals has to be done. Hence we prove the existence and uniqueness for such FBSDEs in the case of a sufficiently small Lipschitz constant of σ with respect to z (see Proposition 6.4), we give L^p-estimates for the solution (see Proposition 6.5) and we also establish a new general comparison result for such fully coupled FBSDEs (Theorem 6.2). We also emphasize that when

σ depends on z it makes the stochastic control much more complicate and the associated HJB equation is combined with an algebraic equation, which is inspired by Wu and Yu [19]. We use the continuation method combined with the fixed point theorem to prove that the algebraic equation has a unique solution whose representation is given (Proposition 4.1).

Let us be more precise. We study a stochastic control problem of fully coupled FBSDE. The cost functional is introduced by the following fully coupled FBSDE:

$$\left\{\begin{array}{l} dX_s^{t,x;u} = b(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)ds + \sigma(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)dB_s, \\ dY_s^{t,x;u} = -f(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)ds + Z_s^{t,x;u}dB_s, \qquad s\in[t,T], \\ X_t^{t,x;u} = x,\ Y_T^{t,x;u} = \Phi(X_T^{t,x;u}), \end{array}\right. \tag{1.1}$$

where $T>0$ is an arbitrarily fixed finite time horizon, $B=(B_s)_{s\in[0,T]}$ is a d-dimensional standard Brownian motion, and $u=(u_s)_{s\in[t,T]}$ is an admissible control. Precise assumptions on the coefficients $b,\ \sigma,\ f,\ \Phi$ are given in the next section. Under our assumptions, (1.1) has a unique solution $(X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u})_{s\in[t,T]}$ and the cost functional is defined by

$$J(t,x;u) = Y_t^{t,x;u}. \tag{1.2}$$

We define the value function of our stochastic control problems as follows:

$$W(t,x) := \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u). \tag{1.3}$$

The objective of our paper is to investigate this value function. The main results of the paper state that W is, deterministic (Proposition 3.1), continuous viscosity solution of the associated HJB equations (Theorem 4.1 and Theorem 4.2). The associated HJB equations are very complicated, we consider two cases of σ for the existence of a viscosity solution.

Case 1. σ does not depend on z, but depends on u.

The associated HJB equation is then the following:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}W(t,x) + H(t,x,W(t,x),DW(t,x),D^2W(t,x)) = 0, \qquad (t,x)\in[0,T)\times\mathbb{R}^n, \\ W(T,x) = \Phi(x), \qquad x\in\mathbb{R}^n, \end{array}\right. \tag{1.4}$$

with

$$H(t,x,y,p,X) = \sup_{u\in U}\{p.b(t,x,y,p.\sigma,u) + \tfrac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(t,x,y,u)X) + f(t,x,y,p.\sigma,u)\},$$

where $t\in[0,T],\ x\in\mathbb{R}^n,\ y\in\mathbb{R},\ p\in\mathbb{R}^n$, and $X\in\mathbb{S}^n$ ($\mathbb{S}^n$ denotes the set of $n\times n$ symmetric matrices).

Case 2. σ does not depend on u, but depends on z.

This case is more complicate than the former one. The associated HJB equation is combined with an algebraic equation as follows:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}W(t,x) + H(t,x,W(t,x),V(t,x)) = 0, \\ V(t,x) = DW(t,x).\sigma(t,x,W(t,x),V(t,x)), \qquad (t,x)\in[0,T)\times\mathbb{R}^n, \\ W(T,x) = \Phi(x), \qquad x\in\mathbb{R}^n. \end{array}\right. \tag{1.5}$$

In this case

$$\begin{aligned} H(t,x,W(t,x),V(t,x)) = \sup_{u\in U}\{&DW(t,x).b(t,x,W(t,x),V(t,x),u) \\ &+\tfrac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(t,x,W(t,x),V(t,x))D^2W(t,x)) + f(t,x,W(t,x),V(t,x),u)\}, \end{aligned}$$

where $t\in[0,T], x\in\mathbb{R}^n$.

The Case 2 is more complicate, the associated HJB equation is combined with an algebraic equation, which is inspired by Wu and Yu [19], [20], we use a new method-the continuation method combined with the fixed point theorem in order to prove for the first time that the algebraic equation has a unique solution, and give the representation for the solution (see Proposition 4.1) which makes that other proofs are available. But

both cases require new estimates and new generalized comparison theorem for small time interval FBSDEs, which are discussed in the Appendix.

Our paper is organized as follows: Section 2 recalls some elements of the theory of fully coupled FBSDEs which will be used in what follows. Section 3 introduces the setting of the stochastic control problems. We prove that the value function W is a deterministic function (Proposition 3.1) which is Lipschitz in x (Lemma 3.2), monotonic (Lemma 3.3) and continuous in t (Theorem 3.2). Moreover, it satisfies the DPP (Theorem 3.1). In Section 4, by using the DPP we prove that W is a viscosity solution of the associated HJB equation in the two cases (Theorem 4.1 and Theorem 4.2) described above. In Section 5, we give two examples. In Appendix we prove some basic important estimates for fully coupled FBSDEs under the monotonic assumptions, and new estimates and new generalized comparison theorem for small time interval FBSDEs.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be the Wiener space, where Ω is the set of continuous functions from $[0,T]$ to $\mathbb{R}^d$ starting from 0 ($\Omega = C_0([0,T];\mathbb{R}^d)$), $\mathcal{F}$ the completed Borel σ-algebra over Ω, and P the Wiener measure. Let B be the canonical process: $B_s(\omega) = \omega_s,\ s \in [0,T],\ \omega \in \Omega$. We denote by $\mathbb{F} = \{\mathcal{F}_s,\ 0 \le s \le T\}$ the natural filtration generated by $\{B_t\}_{t\ge 0}$ and augmented by all P-null sets, i.e.,

$$\mathcal{F}_s = \sigma\{B_r, r \le s\} \vee \mathcal{N}_P, \qquad s \in [0,T],$$

where $\mathcal{N}_P$ is the set of all P-null subsets and T is a fixed real time horizon. We introduce the following two spaces of processes which will be used frequently: for $t_0 \in [0,T]$,
$\mathcal{S}^2(t_0,T;\mathbb{R}^n)$ is the set of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted continuous process $(\psi_t)_{t_0\le t\le T}$ with $E[\sup\limits_{t_0\le t\le T} |\psi_t|^2] < +\infty$;
$\mathcal{H}^2(t_0,T;\mathbb{R}^n)$ is the set of $\mathbb{R}^n$-valued $\mathbb{F}$-progressively meas. process $(\psi_t)_{t_0\le t\le T}$ with $E[\int_{t_0}^T |\psi_t|^2 dt] < +\infty$.

Now we introduce the following fully coupled FBSDE associated with (b,σ,f,ζ,Φ)

$$\begin{cases} dX_s = b(s,X_s,Y_s,Z_s)ds + \sigma(s,X_s,Y_s,Z_s)dB_s, \\ dY_s = -f(s,X_s,Y_s,Z_s)ds + Z_s dB_s, \quad s \in [t,T], \\ X_t = \zeta,\ Y_T = \Phi(X_T), \end{cases} \tag{2.1}$$

where $(X,Y,Z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d},\ T > 0$,
$b : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^n$, $\qquad \sigma : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^{n\times d}$,
$f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$, $\qquad \Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$,
$(b(t,x,y,z))_{t\in[0,T]},\ (\sigma(t,x,y,z))_{t\in[0,T]},\ (f(t,x,y,z))_{t\in[0,T]}$ are $\mathbb{F}$-progressively measurable for each $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, and $\Phi(x)$ is $\mathcal{F}_T$-measurable for each $x \in \mathbb{R}^n$. In this paper we use the usual inner product and the Euclidean norm in $\mathbb{R}^n,\ \mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively. Given an $m \times n$ full-rank matrix G, we define:

$$\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A(t,\lambda) = \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix}(t,\lambda),$$

where G^T is the transposed matrix of G.

We assume that

(B1) (i) $A(t,\lambda)$ is uniformly Lipschitz with respect to λ, and for any λ, $A(\cdot,\lambda) \in \mathcal{H}^2(0,T;\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d})$;
(ii) $\Phi(x)$ is uniformly Lipschitz with respect to $x \in \mathbb{R}^n$, and for any $x \in \mathbb{R}^n$, $\Phi(x) \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^m)$.

The following monotonicity conditions are also necessary:

(B2) (i) $\langle A(t,\lambda) - A(t,\overline{\lambda}), \lambda - \overline{\lambda}\rangle \le -\beta_1 |G\widehat{x}|^2 - \beta_2(|G^T\widehat{y}|^2 + |G^T\widehat{z}|^2)$,
(ii) $\langle \Phi(x) - \Phi(\overline{x}), G(x - \overline{x})\rangle \ge \mu_1 |G\widehat{x}|^2,\quad \widehat{x} = x - \overline{x},\ \widehat{y} = y - \overline{y},\ \widehat{z} = z - \overline{z}$,

where $\beta_1,\ \beta_2,\ \mu_1$ are nonnegative constants with $\beta_1 + \beta_2 > 0,\ \beta_2 + \mu_1 > 0$. Moreover, we have $\beta_1 > 0,\ \mu_1 > 0$ (resp., $\beta_2 > 0$), when $m > n$ (resp., $m < n$).

Remark 2.1. (B2)'-(ii) $\langle \Phi(x)-\Phi(\overline{x}), G(x-\overline{x})\rangle \geq 0$.

When Φ does not depend on x, i.e., $\Phi(x)=\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^m)$, the monotonicity condition (B2)-(i) can be weakened as follows
(B3) (i) $\langle A(t,\lambda)-A(t,\overline{\lambda}), \lambda-\overline{\lambda}\rangle \leq -\beta_1|G\widehat{x}|^2-\beta_2|G^T\widehat{y}|^2,\quad \widehat{x}=x-\overline{x},\ \widehat{y}=y-\overline{y},$
where $\beta_1,\ \beta_2$ are nonnegative constants with $\beta_1+\beta_2>0$. Moreover, we have $\beta_1>0$ (resp., $\beta_2>0$), when $m>n$ (resp., $m<n$).

Lemma 2.1. *Under the assumptions (B1) and (B2), for any initial state $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, FBSDE (2.2) associated with (b,σ,f,ζ,Φ) has a unique adapted solution $(X_s,Y_s,Z_s)_{s\in[t,T]}\in\mathcal{S}^2(t,T;\mathbb{R}^n)\times\mathcal{S}^2(t,T;\mathbb{R}^m)\times\mathcal{H}^2(t,T;\mathbb{R}^{m\times d})$.*

When Φ does not depend on x, i.e., $\Phi(x)=\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^m)$, there is a corresponding result for FBSDE (2.2).

Lemma 2.2. *Under the assumptions (B1) and (B3), for any initial state $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$ and the terminal condition $\Phi(x)=\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^m)$, FBSDE (2.2) associated with (b,σ,f,ζ,ξ) has a unique adapted solution $(X_s,Y_s,Z_s)_{s\in[t,T]}\in\mathcal{S}^2(t,T;\mathbb{R}^n)\times\mathcal{S}^2(t,T;\mathbb{R}^m)\times\mathcal{H}^2(t,T;\mathbb{R}^{m\times d})$.*

The reader can find the proofs of the Lemmas 2.1 and 2.2 in Peng and Wu [17].
Now we give the comparison theorem for FBSDEs which will be used in the later section.

Lemma 2.3. *(Comparison Theorem) Let $m=1$ and assume that (b,σ,f,a,Φ^i), for $i=1,2$, satisfy (B1) and (B2), where $a\in\mathbb{R}^n$ is the initial state for SDE. Let $(X_s^i,Y_s^i,Z_s^i)_{t\leq s\leq T}$ be the solution of FBSDE (2.2) associated with (b,σ,f,a,Φ^i), respectively. If $\Phi^1(x)\geq\Phi^2(x)$, P-a.s. for all $x\in\mathbb{R}^n$. Then, $Y_t^1\geq Y_t^2$, P-a.s.*

As a special case, when $\Phi^1(x)\equiv\xi^1,\ \Phi^2(x)\equiv\xi^2$ and $\xi^1\geq\xi^2$, we have

Lemma 2.4. *Let $m=1$ and assume that (b,σ,f,a,ξ^i), for $i=1,2$, satisfy (B1) and (B3), where $a\in\mathbb{R}^n$ is the initial state for SDE, and $\xi^1,\xi^2\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R})$ are the terminal conditions for related BSDEs, respectively. Let $(X_s^i,Y_s^i,Z_s^i)_{t\leq s\leq T}$ be the solution of FBSDE (2.2) associated with (b,σ,f,a,ξ^i). If $\xi^1\geq\xi^2$, P-a.s. Then, $Y_t^1\geq Y_t^2$, P-a.s.*

The Lemmas 2.3 and 2.4 can be found in Wu [18].

3 A DPP for stochastic optimal control problems of FBSDEs

In this section, we prove the DPP for fully coupled FBSDEs. First we introduce the background of stochastic optimal control problems. We suppose that the control state space U is a compact metric space. $\mathcal{U}$ is the set of all U-valued $\mathbb{F}$-progressively measurable processes. If $u\in\mathcal{U}$, we call u an admissible control.

For a given admissible control $u(\cdot)\in\mathcal{U}$, we regard t as the initial time and $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$ as the initial state. We consider the following fully coupled forward-backward stochastic control system

$$\left\{\begin{array}{l} dX_s^{t,\zeta;u}=b(s,X_s^{t,\zeta;u},Y_s^{t,\zeta;u},Z_s^{t,\zeta;u},u_s)ds+\sigma(s,X_s^{t,\zeta;u},Y_s^{t,\zeta;u},Z_s^{t,\zeta;u},u_s)dB_s,\\ dY_s^{t,\zeta;u}=-f(s,X_s^{t,\zeta;u},Y_s^{t,\zeta;u},Z_s^{t,\zeta;u},u_s)ds+Z_s^{t,\zeta;u}dB_s,\qquad s\in[t,T],\\ X_t^{t,\zeta;u}=\zeta,\ Y_T^{t,\zeta;u}=\Phi(X_T^{t,\zeta;u}),\end{array}\right. \tag{3.1}$$

where the deterministic mappings
$b:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U\to\mathbb{R}^n,\qquad \sigma:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U\to\mathbb{R}^{n\times d},$
$f:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U\to\mathbb{R},\qquad \Phi:\mathbb{R}^n\to\mathbb{R}$
are continuous to (t,u), and satisfy the assumptions (B1) and (B2), for each $u\in U$, and also
(B4) there exists a constant $K\geq 0$ such that, for all $t\in[0,T]$, $u\in U$, $x_1,x_2\in\mathbb{R}^n$, $y_1,y_2\in\mathbb{R}$, $z_1,z_2\in\mathbb{R}^d$,

$$|l(t,x_1,y_1,z_1,u)-l(t,x_2,y_2,z_2,u)|\leq K(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

$l=b,\ \sigma,\ f$, respectively, and $|\Phi(x_1)-\Phi(x_2)|\leq K|x_1-x_2|$.

Remark 3.1. *Under our assumptions, it is obvious that there exists a constant $C \geq 0$ such that,*

$$|b(t,x,y,z,u)| + |\sigma(t,x,y,z,u)| + |f(t,x,y,z,u)| + |\Phi(x)| \leq C(1 + |x| + |y| + |z|),$$

for all $(t,x,y,z,u) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U$. Also notice that now (B4) implies (B1).

Hence, for any $u(\cdot) \in \mathcal{U}$, from Lemma 2.1, FBSDE (3.1) has a unique solution.

From Proposition 6.1 in Appendix, there exists $C \in \mathbb{R}^+$ such that, for any $t \in [0,T]$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}$, we have, P-a.s.:

$$\begin{aligned}
&\text{(i)} E[\sup_{t\leq s\leq T} |X_s^{t,\zeta;u} - X_s^{t,\zeta';u}|^2 + \sup_{t\leq s\leq T} |Y_s^{t,\zeta;u} - Y_s^{t,\zeta';u}|^2 + \int_t^T |Z_s^{t,\zeta;u} - Z_s^{t,\zeta';u}|^2 ds \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2,\\
&\text{(ii)} E[\sup_{t\leq s\leq T} |X_s^{t,\zeta;u}|^2 + \sup_{t\leq s\leq T} |Y_s^{t,\zeta;u}|^2 + \int_t^T |Z_s^{t,\zeta;u}|^2 ds \mid \mathcal{F}_t] \leq C(1 + |\zeta|^2).
\end{aligned} \tag{3.2}$$

Therefore, we get

$$\begin{aligned}
&\text{(i)} \quad |Y_t^{t,\zeta;u}| \leq C(1 + |\zeta|), \text{ P-a.s.;}\\
&\text{(ii)} \quad |Y_t^{t,\zeta;u} - Y_t^{t,\zeta';u}| \leq C|\zeta - \zeta'|, \text{ P-a.s.}
\end{aligned} \tag{3.3}$$

We now introduce the subspaces of admissible controls. An admissible control process $u = (u_r)_{r\in[t,s]}$ on $[t,s]$ is an $\mathbb{F}$-progressively measurable, U-valued process. The set of all admissible controls on $[t,s]$ is denote $\mathcal{U}_{t,s}, \ t \leq s \leq T$.

For a given process $u(\cdot) \in \mathcal{U}_{t,T}$, we define the associated cost functional as follows:

$$J(t,x;u) := Y_s^{t,x;u} \mid_{s=t}, \quad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.4}$$

where the process $Y^{t,x;u}$ is defined by FBSDE (3.1).

From Theorem 6.1 we have, for any $t \in [0,T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,

$$J(t,\zeta;u) = Y_t^{t,\zeta;u}, \text{ P-a.s.} \tag{3.5}$$

For $\zeta = x \in \mathbb{R}^n$, we define the value function as

$$W(t,x) := \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u). \tag{3.6}$$

Remark 3.2. *Thanks to the assumptions (B1) and (B2), the value function $W(t,x)$ is well defined and it is a bounded $\mathcal{F}_t$-measurable random variable. But it turns out to be deterministic.*

Inspired by the method in Buckdahn and Li [2], we can prove that W is deterministic.

Proposition 3.1. *We assume the assumptions (B1) and (B2) hold. Then, for any $(t,x) \in [0,T] \times \mathbb{R}^n$, $W(t,x)$ is a deterministic function in the sense that $W(t,x) = E[W(t,x)]$, P-a.s.*

Proof. Let H denote the Cameron-Martin space of all absolutely continuous elements $h \in \Omega$ whose derivative $\dot{h}$ belongs to $L^2([0,T]; \mathbb{R}^d)$.

For any $h \in H$, we define the mapping $\tau_h \omega := \omega + h, \ \omega \in \Omega$. It is easy to check that $\tau_h : \Omega \to \Omega$ is a bijection, and its law is given by $P \circ [\tau_h]^{-1} = \exp\{\int_0^T \dot{h}_s dB_s - \frac{1}{2}\int_0^T |\dot{h}_s|^2 ds\}P$. For any fixed $(t,x) \in [0,T] \times \mathbb{R}^n$, set $H_t = \{h \in H | h(\cdot) = h(\cdot \wedge t)\}$. The proof can be separated into the following three steps:

(1). For all $u \in \mathcal{U}_{t,T}, \ h \in H_t, \ J(t,x;u)(\tau_h) = J(t,x;u(\tau_h))$, P-a.s.

In fact, using the Girsanov transformation to FBSDE (3.1) (with $\zeta = x$) and comparing the obtained equation with the FBSDE obtained from (3.1) by replacing the transformed control process $u(\tau_h)$ for u, due to the uniqueness of the solution of (3.1) we obtain

$$\begin{aligned}
X_s^{t,x;u}(\tau_h) &= X_s^{t,x;u(\tau_h)}, \text{ for any } s \in [t,T], \text{ P-a.s.,}\\
Y_s^{t,x;u}(\tau_h) &= Y_s^{t,x;u(\tau_h)}, \text{ for any } s \in [t,T], \text{ P-a.s.,}\\
Z_s^{t,x;u}(\tau_h) &= Z_s^{t,x;u(\tau_h)}, \text{ dsdP-a.e. on } [0,T] \times \Omega.
\end{aligned}$$

Hence, $J(t,x;u)(\tau_h)=J(t,x;u(\tau_h)),$ P-a.s.

(2). For any $h\in H_t$, we have

$$\{\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)\}(\tau_h)=\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u)(\tau_h)\},\ \text{P-a.s.}$$

In fact, for convenience, setting $I(t,x)=\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)$, we have $I(t,x)\geq J(t,x;u)$. Then, $I(t,x)(\tau_h)\geq J(t,x;u)(\tau_h),$ P-a.s., for all $u\in\mathcal{U}_{t,T}$. Therefore, $\{\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)\}(\tau_h)\geq \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u)(\tau_h)\}$, P-a.s. On the other hand, for any random variable ξ which satisfies $\xi\geq J(t,x;u)(\tau_h)$, we have $\xi(\tau_{-h})\geq J(t,x;u),$ P-a.s., for all $u\in\mathcal{U}_{t,T}$. So $\xi(\tau_{-h})\geq I(t,x),$ P-a.s., i.e. $\xi\geq I(t,x)(\tau_h),$ P-a.s. Thus, $J(t,x;u)(\tau_h)\geq\{\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)\}(\tau_h),$ P-a.s., for any $u\in\mathcal{U}_{t,T}$. Therefore,

$$\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u)(\tau_h)\}\geq\{\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)\}(\tau_h),\ \text{P-a.s.}$$

From above we get $\{\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)\}(\tau_h)=\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u)(\tau_h)\},$ P-a.s.

(3). Under the Girsanov transformation τ_h, $W(t,x)$ is invariant, i.e.,

$$W(t,x)(\tau_h)=W(t,x),\ \text{P-a.s., for any } h\in H.$$

From the first step and the second one, for all $h\in H_t$, we have

$$\begin{aligned} W(t,x)(\tau_h) &= \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u)(\tau_h)=\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}}\{J(t,x;u)(\tau_h)\}\\ &= \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,T}} J(t,x;u(\tau_h))=W(t,x),\ \text{P-a.s.}\end{aligned}$$

In the latter equality we have used $\{u(\tau_h)\mid u(\cdot)\in\mathcal{U}_{t,T}\}=\mathcal{U}_{t,T}$. Therefore, for any $h\in H_t,\ W(t,x)(\tau_h)=W(t,x),$ P-a.s., and since $W(t,x)$ is $\mathcal{F}_t$-measurable, we have this relation for all $h\in H$.

Combined with the following auxiliary lemma we can complete the proof. □

Lemma 3.1. *Let ζ be a random variable defined over our classical Wiener space $(\Omega,\mathcal{F}_T,P)$, such that $\zeta(\tau_h)=\zeta,$ P-a.s., for any $h\in H$. Then $\zeta=E\zeta,$ P-a.s.*

Its proof can be found in Buckdahn and Li [2].

From (3.3) and (3.6)-the definition of the value function $W(t,x)$, we get the following property:

Lemma 3.2. *There exists a constant $C>0$ such that, for all $0\leq t\leq T,\ x,x'\in\mathbb{R}^n$,*

$$\begin{array}{ll}\text{(i)} & |W(t,x)-W(t,x')|\leq C|x-x'|;\\ \text{(ii)} & |W(t,x)|\leq C(1+|x|).\end{array}\tag{3.7}$$

Lemma 3.3. *Under the assumptions (B1) and (B2), the cost functional $J(t,x;u)$, for any $u\in\mathcal{U}_{t,T}$, and the value function $W(t,x)$ are monotonic in the following sense: for each $x,\bar{x}\in\mathbb{R}^n$, $t\in[0,T]$,*

$$\begin{array}{ll}\text{(i)} & \langle J(t,x;u)-J(t,\bar{x};u),\ G(x-\bar{x})\rangle\geq 0,\ \textit{P-a.s.};\\ \text{(ii)} & \langle W(t,x)-W(t,\bar{x}),\ G(x-\bar{x})\rangle\geq 0.\end{array}$$

Proof. We define $\hat{X}_s=X_s^{t,x;u}-X_s^{t,\bar{x};u},\ \hat{Y}_s=Y_s^{t,x;u}-Y_s^{t,\bar{x};u},\ \hat{Z}_s=Z_s^{t,x;u}-Z_s^{t,\bar{x};u}$, and $\Delta h(s)=h(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)-h(s,X_s^{t,\bar{x};u},Y_s^{t,\bar{x};u},Z_s^{t,\bar{x};u},u_s)$, for $h=b,\ \sigma,\ f,\ A$, respectively.
Applying Itô's formula to $\langle\hat{Y}_s,G\hat{X}_s\rangle$, we get immediately from (B2)

$$\langle J(t,x;u)-J(t,\bar{x};u),G(x-\bar{x})\rangle=E[\langle Y_t^{t,x;u}-Y_t^{t,\bar{x};u},G(x-\bar{x})\rangle\mid\mathcal{F}_t]\geq 0,\quad\text{for any } u\in\mathcal{U}_{t,T}.$$

From the definition of $W(t,x)$, we always have $W(t,x)\geq J(t,x;u),$ P-a.s. for any $u\in\mathcal{U}_{t,T}$. On the other hand, similar to Remark 3.5-(ii), we can get, for any $\varepsilon>0$, the existence of $u^\varepsilon\in\mathcal{U}_{t,T}$, such that $W(t,\bar{x})\leq J(t,\bar{x};u^\varepsilon)+\varepsilon$.
If $G(x-\bar{x})\geq 0$, then $\langle W(t,x)-W(t,\bar{x}),G(x-\bar{x})\rangle\geq(J(t,x;u^\varepsilon)-J(t,\bar{x};u^\varepsilon)-\varepsilon)G(x-\bar{x})\geq-\varepsilon G(x-\bar{x})$.
If $G(x-\bar{x})\leq 0$, then for u^ε such that $W(t,x)\leq J(t,x;u^\varepsilon)+\varepsilon$, $\langle W(t,x)-W(t,\bar{x}),G(x-\bar{x})\rangle\geq-\varepsilon G(\bar{x}-x)$.
Therefore, $\langle W(t,x)-W(t,\bar{x}),G(x-\bar{x})\rangle\geq-\varepsilon|G(x-\overline{x})|$, for any $x,\bar{x}\in\mathbb{R}^n,\ t\in[0,T]$. Consequently, letting $\varepsilon\downarrow 0$, $\langle W(t,x)-W(t,\bar{x}),G(x-\bar{x})\rangle\geq 0$, for any $x,\bar{x}\in\mathbb{R}^n$, $t\in[0,T]$. □

Remark 3.3. *(1) From (B2)-(i) we see that if σ doesn't depend on z, then $\beta_2 = 0$. Furthermore, we assume that:*

(B5) *the Lipschitz constant $L_\sigma \geq 0$ of σ with respect to z is sufficiently small, i.e., there exists some $L_\sigma \geq 0$ small enough such that, for all $t \in [0,T],\ u \in U,\ x_1, x_2 \in \mathbb{R}^n,\ y_1, y_2 \in \mathbb{R},\ z_1, z_2 \in \mathbb{R}^d$, $|\sigma(t,x_1,y_1,z_1,u) - \sigma(t,x_2,y_2,z_2,u)| \leq K(|x_1 - x_2| + |y_1 - y_2|) + L_\sigma |z_1 - z_2|$.*

(2) On the other hand, notice that when σ doesn't depend on z it's obvious that (B5) always holds true.

The notation of stochastic backward semigroup was first introduced by Peng [16] and was applied to prove the DPP for stochastic control problems. Now we discuss a generalized DPP for our stochastic optimal control problem (3.1), (3.6). For this we have to adopt Peng's notion of stochastic backward semigroup, and to define the family of (backward) semigroups associated with FBSDE (3.1).

For given initial data (t,x), a real number $\delta \in (0, T-t]$, an admissible control process $u(\cdot) \in \mathcal{U}_{t,t+\delta}$ and a real-valued random function $\Psi : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{F}_{t+\delta} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable such that (B2)-(ii) holds, we put

$$G^{t,x;u}_{s,t+\delta}[\Psi(t+\delta, \widetilde{X}^{t,x;u}_{t+\delta})] := \widetilde{Y}^{t,x;u}_s,\ \ s \in [t, t+\delta],$$

where $(\widetilde{X}^{t,x;u}_s, \widetilde{Y}^{t,x;u}_s, \widetilde{Z}^{t,x;u}_s)_{t \leq s \leq t+\delta}$ is the solution of the following FBSDE with the time horizon $t+\delta$:

$$\left\{\begin{array}{l} d\widetilde{X}^{t,x;u}_s = b(s, \widetilde{X}^{t,x;u}_s, \widetilde{Y}^{t,x;u}_s, \widetilde{Z}^{t,x;u}_s, u_s)ds + \sigma(s, \widetilde{X}^{t,x;u}_s, \widetilde{Y}^{t,x;u}_s, \widetilde{Z}^{t,x;u}_s, u_s)dB_s, \\ d\widetilde{Y}^{t,x;u}_s = -f(s, \widetilde{X}^{t,x;u}_s, \widetilde{Y}^{t,x;u}_s, \widetilde{Z}^{t,x;u}_s, u_s)ds + \widetilde{Z}^{t,x;u}_s dB_s, \qquad s \in [t, t+\delta], \\ \widetilde{X}^{t,x;u}_t = x,\ \ \widetilde{Y}^{t,x;u}_{t+\delta} = \Psi(t+\delta, \widetilde{X}^{t,x;u}_{t+\delta}). \end{array}\right. \tag{3.8}$$

Remark 3.4. *(1) From Lemma 2.1 and Lemma 2.2 if Ψ doesn't depend on x, we know FBSDE (3.8) has a unique solution $(\widetilde{X}^{t,x;u}, \widetilde{Y}^{t,x;u}, \widetilde{Z}^{t,x;u})$.*
(2) We also point out that if Ψ is Lipschitz with respect to x, FBSDE (3.8) can be also solved under the assumptions (B4) and (B5) on the small interval $[t, t+\delta]$, for any $0 \leq \delta \leq \delta_0$, where small enough $\delta_0 > 0$ is independent of (t,x) and the control u, from Proposition 6.4.

Since Φ satisfies (B2)-(ii) the solution $(X^{t,x;u}, Y^{t,x;u}, Z^{t,x;u})$ of FBSDE (3.1) exists and we get

$$G^{t,x;u}_{t,T}[\Phi(X^{t,x;u}_T)] = G^{t,x;u}_{t,t+\delta}[Y^{t,x;u}_{t+\delta}].$$

Moreover, we have

$$J(t,x;u) = Y^{t,x;u}_t = G^{t,x;u}_{t,T}[\Phi(X^{t,x;u}_T)] = G^{t,x;u}_{t,t+\delta}[Y^{t,x;u}_{t+\delta}] = G^{t,x;u}_{t,t+\delta}[J(t+\delta, X^{t,x;u}_{t+\delta}; u)]. \tag{3.9}$$

Theorem 3.1. *Under the assumptions (B2), (B4) and (B5), the value function $W(t,x)$ satisfies the following DPP: there exists sufficiently small $\delta_0 > 0$, such that for any $0 \leq \delta \leq \delta_0,\ t \in [0, T-\delta],\ x \in \mathbb{R}^n$,*

$$W(t,x) = \operatorname{ess\,sup}_{u \in \mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u}_{t+\delta})].$$

Proof. With the help of Lemma 3.2, (3.9), Theorem 5.2, Corollary 5.1, and Proposition 5.4, adapting the method of the proof of Theorem 3.6 in [2], we can complete the proof. □

From its proof we can get

Remark 3.5. (i) *For all $u \in \mathcal{U}_{t,t+\delta}$,*

$$W(t,x) (= W_\delta(t,x)) \geq G^{t,x;u}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u}_{t+\delta})], \quad P\text{-}a.s.$$

(ii) *For any $(t,x) \in [0,T] \times \mathbb{R}^n$, $\delta \in [0, \delta_0]$ and $\varepsilon > 0$, there exists some $u^\varepsilon(\cdot) \in \mathcal{U}_{t,t+\delta}$ such that*

$$W(t,x) (= W_\delta(t,x)) \leq G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x;u^\varepsilon}_{t+\delta})] + \varepsilon,\ \ P\text{-}a.s.$$

Notice that from the definition of our stochastic backward semigroup we know here

$$G^{t,x;u}_{s,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u}_{t+\delta})]=\widetilde{Y}^{t,x;u}_s,\ \ s\in[t,t+\delta],\ \ u(\cdot)\in\mathcal{U}_{t,t+\delta},$$

where $(\widetilde{X}^{t,x;u}_s,\widetilde{Y}^{t,x;u}_s,\widetilde{Z}^{t,x;u}_s)_{t\le s\le t+\delta}$ is the solution of the following FBSDE with the time horizon $t+\delta$:

$$\left\{\begin{array}{l} d\widetilde{X}^{t,x;u}_s=b(s,\widetilde{X}^{t,x;u}_s,\widetilde{Y}^{t,x;u}_s,\widetilde{Z}^{t,x;u}_s,u_s)ds+\sigma(s,\widetilde{X}^{t,x;u}_s,\widetilde{Y}^{t,x;u}_s,\widetilde{Z}^{t,x;u}_s,u_s)dB_s,\\ d\widetilde{Y}^{t,x;u}_s=-f(s,\widetilde{X}^{t,x;u}_s,\widetilde{Y}^{t,x;u}_s,\widetilde{Z}^{t,x;u}_s,u_s)ds+\widetilde{Z}^{t,x;u}_s dB_s,\qquad s\in[t,t+\delta],\\ \widetilde{X}^{t,x;u}_t=x,\ \ \widetilde{Y}^{t,x;u}_{t+\delta}=W(t+\delta,\widetilde{X}^{t,x;u}_{t+\delta}). \end{array}\right. \tag{3.10}$$

Due to Proposition 6.4 there exists sufficiently small $\delta_0>0$, such that for any $0\le\delta\le\delta_0$, the above equation (3.10) has a unique solution $(\widetilde{X}^{t,x;u},\widetilde{Y}^{t,x;u},\widetilde{Z}^{t,x;u})$ on the time interval $[t,t+\delta]$.

From Lemma 3.2, we get the value function $W(t,x)$ is Lipschitz continuous in x, uniformly in t. Now we can get the continuity property of $W(t,x)$ in t with the help of Theorem 3.1.

Theorem 3.2. *Under (B2), (B4) and (B5), the value function $W(t,x)$ is continuous in t.*

Proof. Let $(t,x)\in[0,T]\times\mathbb{R}^n$. In order to obtain W is continuous in t, it is sufficient to prove the following inequality: there exists some constant C, such that

$$-C(1+|x|)\delta^{\frac12}\le W(t,x)-W(t+\delta,x)\le C(1+|x|)\delta^{\frac12},\ \text{for all } 0\le\delta\le T-t \text{ sufficiently small.}$$

We will only prove the second inequality, the proof of the first one is similar.

From Remark 3.5, there exists $u^\varepsilon\in\mathcal{U}$, such that

$$G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon}_{t+\delta})]+\varepsilon\ge W(t,x)\ge G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon}_{t+\delta})].$$

Therefore, $W(t,x)-W(t+\delta,x)\le G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon}_{t+\delta})]+\varepsilon-W(t+\delta,x)=I^1_\delta+I^2_\delta+\varepsilon$, where

$$\begin{array}{rcl} I^1_\delta &=& G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon}_{t+\delta})]-G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,x)],\\ I^2_\delta &=& G^{t,x;u^\varepsilon}_{t,t+\delta}[W(t+\delta,x)]-W(t+\delta,x). \end{array}$$

Notice also that $G^{t,x;u}_{s,t+\delta}[W(t+\delta,x)]=\hat{Y}^{t,x;u}_s,\ s\in[t,t+\delta],\ u(\cdot)\in\mathcal{U}_{t,t+\delta}$, where $(\hat{X}^{t,x;u}_s,\hat{Y}^{t,x;u}_s,\hat{Z}^{t,x;u}_s)_{t\le s\le t+\delta}$ is the solution of the following FBSDE with the time horizon $t+\delta$:

$$\left\{\begin{array}{l} d\hat{X}^{t,x;u}_s=b(s,\hat{X}^{t,x;u}_s,\hat{Y}^{t,x;u}_s,\hat{Z}^{t,x;u}_s,u_s)ds+\sigma(s,\hat{X}^{t,x;u}_s,\hat{Y}^{t,x;u}_s,\hat{Z}^{t,x;u}_s,u_s)dB_s,\\ d\hat{Y}^{t,x;u}_s=-f(s,\hat{X}^{t,x;u}_s,\hat{Y}^{t,x;u}_s,\hat{Z}^{t,x;u}_s,u_s)ds+\hat{Z}^{t,x;u}_s dB_s,\qquad s\in[t,t+\delta],\\ \hat{X}^{t,x;u}_t=x,\ \ \hat{Y}^{t,x;u}_{t+\delta}=W(t+\delta,x). \end{array}\right. \tag{3.11}$$

Applying Itô's formula to $e^{\beta s}|\widetilde{Y}^{t,x;u^\varepsilon}_s-\hat{Y}^{t,x;u^\varepsilon}_s|^2$, by taking β large enough and using standard methods for BSDEs, we get with the help of (3.7) and Proposition 6.5-(ii) for equations (3.10) and (3.11) that

$$\begin{array}{rl} & |\widetilde{Y}^{t,x;u^\varepsilon}_t-\hat{Y}^{t,x;u^\varepsilon}_t|^2\\ \le & CE[|W(t+\delta,\widetilde{X}^{t,x;u^\varepsilon}_{t+\delta})-W(t+\delta,x)|^2\mid\mathcal{F}_t]+CE[\int_t^{t+\delta}|\widetilde{X}^{t,x;u^\varepsilon}_r-\hat{X}^{t,x;u^\varepsilon}_r|^2dr\mid\mathcal{F}_t]\\ \le & CE[|\widetilde{X}^{t,x;u^\varepsilon}_{t+\delta}-x|^2\mid\mathcal{F}_t]+C\delta(E[\sup\limits_{t\le r\le t+\delta}|\widetilde{X}^{t,x;u^\varepsilon}_r-x|^2\mid\mathcal{F}_t]+E[\sup\limits_{t\le r\le t+\delta}|\hat{X}^{t,x;u^\varepsilon}_r-x|^2\mid\mathcal{F}_t])\\ \le & C\delta(1+|x|^2),\ \text{P-a.s.} \end{array} \tag{3.12}$$

That is, there exists some constant C independent of the controls such that

$$|I^1_\delta|=|\widetilde{Y}^{t,x;u^\varepsilon}_t-\hat{Y}^{t,x;u^\varepsilon}_t|\le C(1+|x|)\delta^{\frac12},\ \text{P-a.s.}$$

From equation (3.11), Remark 3.1, and Proposition 6.5-(i) (the estimates for FBSDE (3.11))

$$\begin{array}{rcl} |I^2_\delta| &=& |E[W(t+\delta,x)+\int_t^{t+\delta}f(s,\hat{X}^{t,x;u^\varepsilon}_s,\hat{Y}^{t,x;u^\varepsilon}_s,\hat{Z}^{t,x;u^\varepsilon}_s,u^\varepsilon_s)ds\mid\mathcal{F}_t]-W(t+\delta,x)|\\ &\le& C\delta^{\frac12}E[\int_t^{t+\delta}(1+|\hat{X}^{t,x;u^\varepsilon}_s|+|\hat{Y}^{t,x;u^\varepsilon}_s|+|\hat{Z}^{t,x;u^\varepsilon}_s|)^2ds\mid\mathcal{F}_t]^{\frac12}\\ &\le& C(1+|x|)\delta^{\frac12}. \end{array}$$

Therefore, $W(t,x)-W(t+\delta,x)\le C(1+|x|)\delta^{\frac12}+\varepsilon$. Letting $\varepsilon\downarrow0$, we complete the proof. □

4 Viscosity solutions of HJB equations

In this section we show that the value function $W(t,x)$ defined in (3.6) is a viscosity solution of the corresponding HJB equation. For this we use Peng's BSDE approach [16] developed from stochastic control problems of decoupled FBSDEs, but still more difficulties for two cases, especially for Case 2.

Case 1. We suppose that σ does not depend on z, but depends on u.

Then the equation (3.1) becomes the following equation (4.1):

$$\left\{\begin{array}{l} dX_s^{t,x;u}=b(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)ds+\sigma(s,X_s^{t,x;u},Y_s^{t,x;u},u_s)dB_s,\\ dY_s^{t,x;u}=-f(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)ds+Z_s^{t,x;u}dB_s,\quad s\in[t,T],\\ X_t^{t,x;u}=x,\ Y_T^{t,x;u}=\Phi(X_T^{t,x;u}).\end{array}\right. \tag{4.1}$$

We consider the following HJB equation:

$$\left\{\begin{array}{l} \frac{\partial}{\partial t}W(t,x)+H(t,x,W(t,x),DW(t,x),D^2W(t,x))=0,\qquad (t,x)\in[0,T)\times\mathbb{R},\\ W(T,x)=\Phi(x),\qquad x\in\mathbb{R}.\end{array}\right. \tag{4.2}$$

In this case the Hamiltonian is given by

$$H(t,x,y,p,X)=\sup_{u\in U}\{p.b(t,x,y,p.\sigma,u)+\tfrac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(t,x,y,u)X)+f(t,x,y,p.\sigma,u)\},$$

where $t\in[0,T],\ x\in\mathbb{R},\ y\in\mathbb{R},\ p\in\mathbb{R}$, and $X\in\mathbb{R}$.

Let us first recall the definition of a viscosity solution of equation (4.2). More details on viscosity solutions can be found in Crandall, Ishill and Lions [3].

Definition 4.1. *A real-valued continuous function $W\in C([0,T]\times\mathbb{R}^k)$ is called*
(i) *a viscosity subsolution of equation (4.2) if $W(T,x)\le\Phi(x)$, for all $x\in\mathbb{R}^k$, and if for all functions $\varphi\in C^3_{l,b}([0,T]\times\mathbb{R}^k)$ and for all $(t,x)\in[0,T)\times\mathbb{R}^k$ such that $W-\varphi$ attains a local maximum at (t,x),*

$$\frac{\partial\varphi}{\partial t}(t,x)+H(t,x,\varphi,D\varphi,D^2\varphi)\ge 0;$$

(ii) *a viscosity supersolution of equation (4.2) if $W(T,x)\ge\Phi(x)$, for all $x\in\mathbb{R}^k$, and if for all functions $\varphi\in C^3_{l,b}([0,T]\times\mathbb{R}^k)$ and for all $(t,x)\in[0,T)\times\mathbb{R}^k$ such that $W-\varphi$ attains a local minimum at (t,x),*

$$\frac{\partial\varphi}{\partial t}(t,x)+H(t,x,\varphi,D\varphi,D^2\varphi)\le 0;$$

(iii) *a viscosity solution of equation (4.2) if it is both a viscosity sub- and supersolution of equation (4.2).*

Remark 4.1. *$C^3_{l,b}([0,T]\times\mathbb{R}^k)$ denotes the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.*

Theorem 4.1. *Under the assumptions (B2) and (B4), the value function $W(t,x)$ defined in (3.6) is a viscosity solution of (4.2).*

Proof. Obviously, $W(T,x)=\Phi(x),\ x\in\mathbb{R}$. Let us show that W is a viscosity subsolution, the proof for the viscosity supersolution is similar. We suppose that $\varphi\in C^3_{l,b}([0,T]\times\mathbb{R})$ and that $(t,x)\in[0,T)\times\mathbb{R}$ is such that $W-\varphi$ attains its maximum at (t,x). Since W is continuous and of at most linear growth, we only need to consider the global maximum at (t,x). Without loss of generality we may assume that $\varphi(t,x)=W(t,x)$. We consider the following equation:

$$\left\{\begin{array}{l} dX_s^u=b(s,X_s^u,Y_s^u,Z_s^u,u_s)ds+\sigma(s,X_s^u,Y_s^u,u_s)dB_s,\\ dY_s^u=-f(s,X_s^u,Y_s^u,Z_s^u,u_s)ds+Z_s^udB_s,\qquad s\in[t,t+\delta],\\ X_t^u=x,\ Y_{t+\delta}^u=\varphi(t+\delta,X_{t+\delta}^u),\quad 0\le\delta\le T-t.\end{array}\right. \tag{4.3}$$

From Propositions 6.4 and 6.5 in Appendix, we know there exists sufficiently small $0 < \delta_1 < T-t$, such that for any $0 \le \delta \le \delta_1$, FBSDE (4.3) has a unique solution $(X^u_s, Y^u_s, Z^u_s)_{t\le s\le t+\delta} \in \mathcal{S}^2(t,t+\delta;\mathbb{R}) \times \mathcal{S}^2(t,t+\delta;\mathbb{R}) \times \mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$, and for $p \ge 2$,

$$\begin{array}{lll} \text{(i)} & E[\sup\limits_{t\le s\le t+\delta}|X^u_s|^p + \sup\limits_{t\le s\le t+\delta}|Y^u_s|^p + (\int_t^{t+\delta}|Z^u_s|^2ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \le C(1+|x|^p), \text{ P-a.s.}; & \\ \text{(ii)} & E[\sup\limits_{t\le s\le t+\delta}|X^u_s - x|^p \mid \mathcal{F}_t] \le C\delta^{\frac{p}{2}}(1+|x|^p), \text{ P-a.s.}; & \\ \text{(iii)} & E[(\int_t^{t+\delta}|Z^u_s|^2ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \le C\delta^{\frac{p}{2}}(1+|x|^p), \text{ P-a.s.} & \end{array} \tag{4.4}$$

From the definition of the backward stochastic semigroup for fully coupled FBSDE, we have

$$G^{t,x;u}_{s,t+\delta}[\varphi(t+\delta, X^u_{t+\delta})] = Y^u_s, \quad s\in[t,t+\delta],\ 0\le\delta\le\delta_1. \tag{4.5}$$

From the DPP (Theorem 3.1), we have

$$\varphi(t,x) = W(t,x) = \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}[W(t+\delta, \widetilde{X}^{t,x,u}_{t+\delta})],\ 0\le\delta\le\delta_1,$$

where $\widetilde{X}^{t,x,u}$ is defined by FBSDE (3.10).
From $W(s,y) \le \varphi(s,y),\ (s,y)\in[0,T)\times\mathbb{R}$, and the monotonicity property of $G^{t,x;u}_{t,t+\delta}[\cdot]$ (see Theorem 6.2 in Appendix) we have,

$$\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}[\varphi(t+\delta, X^u_{t+\delta})] - \varphi(t,x) \ge 0, \quad 0\le\delta\le\delta_1. \tag{4.6}$$

Now we define

$$\begin{array}{rcl} Y^{1,u}_s &=& Y^u_s - \varphi(s,X^u_s) \\ &=& \int_s^{t+\delta} f(r,X^u_r,Y^u_r,Z^u_r,u_r)dr - \int_s^{t+\delta} Z^u_r dB_r + \varphi(t+\delta, X^u_{t+\delta}) - \varphi(s,X^u_s). \end{array} \tag{4.7}$$

Using Itô's formula to $\varphi(s,X^u_s)$, and setting $Z^{1,u}_s = Z^u_s - D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s)$, we have

$$\left\{\begin{array}{rcl} Y^{1,u}_s &=& \int_s^{t+\delta}[\frac{\partial}{\partial r}\varphi(r,X^u_r) + D\varphi(r,X^u_r).b(r,X^u_r,Y^u_r,Z^u_r,u_r) + \frac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(r,X^u_r,Y^u_r,u_r)D^2\varphi(r,X^u_r)) \\ && +f(r,X^u_r,Y^u_r,Z^u_r,u_r)]dr - \int_s^{t+\delta} Z^{1,u}_r dB_r, \\ Z^{1,u}_s &=& Z^u_s - D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s), \quad t\le s\le t+\delta. \end{array}\right. \tag{4.8}$$

From (4.5), (4.6), (4.7), we have

$$\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}} Y^{1,u}_t \ge 0, \text{ P-a.s.} \tag{4.9}$$

For $(s,x,y,z,u)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times U$, we define

$$\begin{array}{rcl} L(s,x,y,z,u) &=& \frac{\partial}{\partial s}\varphi(s,x) + D\varphi(s,x).b(s,x,y+\varphi(s,x),z,u) + \frac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(s,x,y+\varphi(s,x),u)D^2\varphi(s,x)) \\ && +f(s,x,y+\varphi(s,x),z,u), \\ F(s,x,y,z,u) &=& L(s,x,y,z+D\varphi(s,x).\sigma(s,x,y+\varphi(s,x),u),u). \end{array}$$

Then equation (4.8) can be reformulated as:

$$\left\{\begin{array}{rcl} dY^{1,u}_s &=& -F(s,X^u_s,Y^{1,u}_s,Z^{1,u}_s,u_s)ds + Z^{1,u}_s dB_s, \quad s\in[t,t+\delta], \\ Y^{1,u}_{t+\delta} &=& 0, \end{array}\right. \tag{4.10}$$

where $Y^{1,u}_s = Y^u_s - \varphi(s,X^u_s)$, $Z^{1,u}_s = Z^u_s - D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s)$.
Obviously, equation (4.10) has a unique solution $(Y^{1,u}_s, Z^{1,u}_s)_{s\in[t,t+\delta]} \in \mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$. Indeed, equation (4.10) has a solution $(Y^u_s - \varphi(s,X^u_s), Z^u_s - D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s))_{s\in[t,t+\delta]}$. If (4.10) has another solution $(\tilde{Y}^{1,u}_s, \tilde{Z}^{1,u}_s)_{s\in[t,t+\delta]} \in \mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$, then $(X^u_s, \tilde{Y}^{1,u}_s + \varphi(s,X^u_s), \tilde{Z}^{1,u}_s + D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s))_{s\in[t,t+\delta]}$ is the solution of equation (4.3), from the uniqueness of the solution of FBSDE (4.3), we have $\tilde{Y}^{1,u}_s + \varphi(s,X^u_s) = Y^u_s,\ \tilde{Z}^{1,u}_s + D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s) = Z^u_s$, i.e., $\tilde{Y}^{1,u}_s = Y^u_s - \varphi(s,X^u_s)$, P-a.s., $\tilde{Z}^{1,u}_s = Z^u_s - D\varphi(s,X^u_s).\sigma(s,X^u_s,Y^u_s,u_s)$, a.s., a.e., $s\in[t,t+\delta]$.

Now we need to study the following BSDE:

$$\begin{cases} dY_s^{2,u} &= -L(s,x,0,\hat{Z}_s^u,u_s)ds + Z_s^{2,u}dB_s, \quad s\in[t,t+\delta], \\ Y_{t+\delta}^{2,u} &= 0, \end{cases} \tag{4.11}$$

where $\hat{Z}_s^u = Z_s^{1,u} + D\varphi(s,x).\sigma(s,x,Y_s^{1,u}+\varphi(s,x),u_s),\ s\in[t,t+\delta]$. Notice that

$$\begin{array}{ll} \text{(i)} & |L(s,x,y,z,u)-L(s,x,y',z',u)| \le C(1+|x|+|y|+|y'|)(|y-y'|+|z-z'|); \\ \text{(ii)} & |L(s,x,y,z,u)| \le C(1+|x|^2+|y|^2+|z|), \quad \forall (s,x,y,z,u)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U. \end{array} \tag{4.12}$$

Since $\hat{Z}^u\in\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$, and $E[\int_t^{t+\delta}|L(s,x,0,\hat{Z}_s^u,u_s)|^2ds]<+\infty$, from Lemma 2.1 in [2], BSDE (4.11) has a unique solution $(Y_s^{2,u},Z_s^{2,u})_{s\in[t,t+\delta]}\in\mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$.
Now we prove some lemmas for the proof of Theorem 4.1.

Lemma 4.1. *For all $u\in\mathcal{U}_{t,t+\delta}$, we have*

$$E[\int_t^{t+\delta}(|Y_s^{1,u}|+|Z_s^{1,u}|)ds \mid \mathcal{F}_t] \le C\delta^{\frac{5}{4}},\ P\text{-a.s.},\ 0\le\delta\le\delta_1,$$

where the constant C is independent of the control u and of $\delta>0$.

Proof. From (4.7) and (4.4),

$$\begin{array}{rcl} |Y_s^{1,u}| &=& |E[\int_s^{t+\delta}f(r,X_r^u,Y_r^u,Z_r^u,u_r)dr \mid \mathcal{F}_s] + E[\varphi(t+\delta,X_{t+\delta}^u)-\varphi(s,X_s^u) \mid \mathcal{F}_s]| \\ &\le& CE[\int_s^{t+\delta}(1+|X_r^u|+|Y_r^u|+|Z_r^u|)dr \mid \mathcal{F}_s] + C\delta + CE[|X_{t+\delta}^u - X_s^u| \mid \mathcal{F}_s] \\ &\le& C\delta^{\frac12}(E[\int_s^{t+\delta}(1+|X_r^u|^2+|Y_r^u|^2+|Z_r^u|^2)dr \mid \mathcal{F}_s])^{\frac12} + C\delta + C\delta^{\frac12}(1+|X_s^u|) \\ &\le& C\delta^{\frac12}(1+|X_s^u|), \quad \text{P-a.s.},\ s\in[t,t+\delta], \end{array} \tag{4.13}$$

where notice that $(X_s^u,Y_s^u,Z_s^u)=(X_s^{s,X_s^u},Y_s^{s,X_s^u},Z_s^{s,X_s^u})$, P-a.s., $s\in[t,t+\delta]$, from the uniqueness of the solution of FBSDE (4.3) on $[t,t+\delta]$. Furthermore, from (4.7), we get $|Y_s^u|\le C(1+|X_s^u|)$, P-a.s., $s\in[t,t+\delta]$. On the other hand, $Z_s^{1,u}=Z_s^u - D\varphi(s,X_s^u).\sigma(s,X_s^u,Y_s^u,u_s)$, we have

$$|Z_s^{1,u}|\le C(1+|X_s^u|+|Z_s^u|), \quad \text{P-a.s.},\ s\in[t,t+\delta]. \tag{4.14}$$

From (4.10), (4.12), (4.13), (4.14) and (4.4)

$$\begin{array}{rl} & |Y_t^{1,u}|^2 + E[\int_t^{t+\delta}|Z_r^{1,u}|^2dr \mid \mathcal{F}_t] \\ = & 2E[\int_t^{t+\delta}Y_r^{1,u}F(r,X_r^u,Y_r^{1,u},Z_r^{1,u},u_r)dr \mid \mathcal{F}_t] \\ \le & C\delta^{\frac12}E[\int_t^{t+\delta}(1+|X_r^u|^2+|X_r^u|^3)dr \mid \mathcal{F}_t] + C\delta^{\frac12}E[\int_t^{t+\delta}|Z_r^u|^2dr \mid \mathcal{F}_t] \\ \le & C\delta^{\frac32},\ \text{P-a.s.} \end{array} \tag{4.15}$$

Therefore, from (4.13), (4.15) and (4.4)

$$\begin{array}{rl} & E[\int_t^{t+\delta}(|Y_s^{1,u}|+|Z_s^{1,u}|)ds \mid \mathcal{F}_t] \\ \le & C\delta^{\frac12}E[\int_t^{t+\delta}(1+|X_r^u|)dr \mid \mathcal{F}_t] + C\delta^{\frac12}(E[\int_t^{t+\delta}|Z_r^{1,u}|^2dr \mid \mathcal{F}_t])^{\frac12} \\ \le & C\delta^{\frac54},\ \text{P-a.s.},\ 0\le\delta\le\delta_1. \end{array}$$

□

Remark 4.2. *From (4.4), and (4.14),*

$$E[(\int_t^{t+\delta}|Z_r^{1,u}|^2dr)^2 \mid \mathcal{F}_t] \le C\delta^2. \tag{4.16}$$

Lemma 4.2. *For all $u \in \mathcal{U}_{t,t+\delta}$, we have*

$$|Y_t^{1,u} - Y_t^{2,u}| \leq C\delta^{\frac{5}{4}}, \quad P\text{-a.s.}, \ 0 \leq \delta \leq \delta_1,$$

where C is independent of the control process u and of $\delta > 0$.

Proof. We define: $g(s) = L(s, X_s^u, 0, Z_s^u, u_s) - L(s, x, 0, Z_s^u, u_s)$, $\rho_0(r) = (1 + |x|^2 + |Z_r^u|)(r + r^2)$, $r \geq 0$, Then, $|g(s)| \leq C\rho_0(|X_s^u - x|)$, $s \in [t, t+\delta]$. From (4.4), (4.10), (4.11), the definitions of F and L, we have

$$\begin{array}{rcl} |Y_t^{1,u} - Y_t^{2,u}| & = & |E[(Y_t^{1,u} - Y_t^{2,u})|\mathcal{F}_t]| \\ & = & |E[\int_t^{t+\delta}(L(s, X_s^u, Y_s^{1,u}, Z_s^u, u_s) - L(s, x, 0, \hat{Z}_s^u, u_s))ds \mid \mathcal{F}_t]| \\ & \leq & E[\int_t^{t+\delta}[C(1 + |X_s^u| + |Y_s^{1,u}|)|Y_s^{1,u}| + C(1 + |x|)|Z_s^u - \hat{Z}_s^u| + g(s)]ds \mid \mathcal{F}_t]. \end{array} \tag{4.17}$$

Notice that from (4.13), $E[\int_t^{t+\delta}(1 + |X_s^u| + |Y_s^{1,u}|)|Y_s^{1,u}|ds \mid \mathcal{F}_t] \leq C\delta^{\frac{3}{2}}$; from (4.8), (4.11) and (4.4)

$$\begin{array}{rl} & E[\int_t^{t+\delta}|Z_s^u - \hat{Z}_s^u|ds \mid \mathcal{F}_t] \\ = & E[\int_t^{t+\delta}|Z_s^{1,u} + D\varphi(s, X_s^u).\sigma(s, X_s^u, Y_s^{1,u} + \varphi(s, X_s^u), u_s) \\ & \quad -(Z_s^{1,u} + D\varphi(s, x).\sigma(s, x, Y_s^{1,u} + \varphi(s, x), u_s))|ds \mid \mathcal{F}_t] \\ \leq & CE[\int_t^{t+\delta}|X_s^u - x|(1 + |X_s^u| + |Y_s^{1,u}|)ds \mid \mathcal{F}_t] \\ \leq & C\delta E[\sup\limits_{t \leq s \leq t+\delta}|X_s^u - x|(1 + |X_s^u| + |Y_s^{1,u}|)|\mathcal{F}_t] \leq C\delta^{\frac{3}{2}}, \ \text{P-a.s.}; \end{array}$$

furthermore, from (4.4),

$$\begin{array}{rl} & E[\int_t^{t+\delta} g(s)ds \mid \mathcal{F}_t] \leq \delta^{\frac{1}{2}}(E[\int_t^{t+\delta} g(s)^2 ds \mid \mathcal{F}_t])^{\frac{1}{2}} \\ \leq & C\delta^{\frac{1}{2}}(E[\int_t^{t+\delta}(1 + |x|^4 + |Z_r^u|^2)(|X_r^u - x|^2 + |X_r^u - x|^4)dr \mid \mathcal{F}_t])^{\frac{1}{2}} \leq C\delta^{\frac{5}{4}}, \quad \text{P-a.s.} \end{array}$$

From (4.17), the proof is complete. □

Now we consider the following equation:

$$\left\{ \begin{array}{rcl} dY_s^{3,u} & = & -L(s, x, 0, D\varphi(s, x).\sigma(s, x, \varphi(s, x), u_s), u_s)ds + Z_s^{3,u}dB_s, \quad s \in [t, t+\delta], \\ Y_{t+\delta}^{3,u} & = & 0, \end{array} \right. \tag{4.18}$$

where $u(\cdot) \in \mathcal{U}_{t,t+\delta}$. Notice that

$$L(s, x, 0, D\varphi(s, x).\sigma(s, x, \varphi(s, x), u_s), u_s) = F(s, x, 0, 0, u_s).$$

Lemma 4.3. *For all $u \in \mathcal{U}_{t,t+\delta}$, we have*

$$|Y_t^{2,u} - Y_t^{3,u}| \leq C\delta^{\frac{5}{4}}, \quad P\text{-a.s.}, \ 0 \leq \delta \leq \delta_1,$$

where C is independent of the control process u and of $\delta > 0$.

Proof. From (4.11), (4.18), (4.12) and Lemma 4.1,

$$\begin{array}{rcl} |Y_t^{2,u} - Y_t^{3,u}| & = & |E[\int_t^{t+\delta}(L(s, x, 0, \hat{Z}_s^u, u_s) - L(s, x, 0, D\varphi(s, x).\sigma(s, x, \varphi(s, x), u_s), u_s))ds \mid \mathcal{F}_t]| \\ & \leq & CE[\int_t^{t+\delta}(1 + |x|)(|Y_s^{1,u}| + |Z_s^{1,u}|)ds \mid \mathcal{F}_t] \\ & \leq & C\delta^{\frac{5}{4}}, \quad \text{P-a.s.}, \ 0 \leq \delta \leq \delta_1. \end{array}$$

□

Lemma 4.4. *Let $Y_0(\cdot)$ be the solution of the following ordinary differential equation:*

$$\left\{ \begin{array}{rcl} \dot{Y}_0(s) & = & -F_0(s, x, 0, 0), \ s \in [t, t+\delta], \\ Y_0(t+\delta) & = & 0, \end{array} \right. \tag{4.19}$$

where

$$F_0(s, x, 0, 0) = \sup_{u \in U} F(s, x, 0, 0, u). \tag{4.20}$$

Then, P-a.s.,

$$\operatorname{ess\,sup}_{u \in \mathcal{U}_{t,t+\delta}} Y_t^{3,u} = Y_0(t). \tag{4.21}$$

Proof. Obviously, (4.19) has a unique solution. From the definition of $F_0(s,x,0,0)$, we know

$$F_0(s,x,0,0)\geq F(s,x,0,0,u_s),\quad \text{for any } u\in\mathcal{U}_{t,t+\delta}.$$

Therefore, from the comparison theorem of BSDE (see, Lemma 2.2 in [2]),

$$\tilde{Y_0}(s)\geq Y_s^{3,u},\ \text{P-a.s.,}\quad \text{for any } s\in[t,t+\delta],\ \text{for any } u\in\mathcal{U}_{t,t+\delta},$$

where $(\tilde{Y_0}(\cdot),\tilde{Z}_0(\cdot))$ is the solution of the following BSDE:

$$\begin{cases} d\tilde{Y_0}(s) &= -F_0(s,x,0,0)ds+\tilde{Z}_0(s)dB_s,\ s\in[t,t+\delta],\\ \tilde{Y_0}(t+\delta) &= 0.\end{cases}$$

In fact, $(\tilde{Y_0}(s),\tilde{Z}_0(s))=(Y_0(s),0)$. Therefore, $Y_0(t)\geq Y_t^{3,u}$, P-a.s., for any $u\in\mathcal{U}_{t,t+\delta}$.

On the other hand, since $F_0(s,x,0,0)=\sup\limits_{u\in U}F(s,x,0,0,u)$, there exists some measurable function $\tilde{u}(s,x):[t,t+\delta]\times\mathbb{R}^n\to U$, such that $F(s,x,0,0,\tilde{u}(s,x))=F_0(s,x,0,0)$. Define $\tilde{u}_s^0=\tilde{u}(s,x),\ s\in[t,t+\delta]$, then $\tilde{u}^0\in\mathcal{U}_{t,t+\delta}$ and $F_0(s,x,0,0)=F(s,x,0,0,\tilde{u}_s^0),\ s\in[t,t+\delta]$. Consequently, from the uniqueness of the solution of the BSDE it follows that $Y_0(t)=Y_t^{3,\tilde{u}^0}$, P-a.s. Therefore, $\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}Y_t^{3,u}=Y_0(t)$, P-a.s.

□

Now we are able to complete the proof of Theorem 4.1 as follows:

Indeed, from (4.9) we know that $\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}Y_t^{1,u}\geq 0$, P-a.s. Therefore, from the Lemmas 4.2 and 4.3 we get $\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}Y_t^{3,u}\geq -C\delta^{\frac{5}{4}}$, P-a.s. Thus, from Lemma 4.4, $Y_0(t)\geq -C\delta^{\frac{5}{4}}$, where Y_0 is the solution of (4.19). Then,

$$\frac{1}{\delta}\int_t^{t+\delta}F_0(s,x,0,0)ds\geq -C\delta^{\frac{1}{4}},\quad 0\leq\delta\leq\delta_1.$$

It follows by letting $\delta\to 0$ that

$$\sup_{u\in U}F(t,x,0,0,u)=F_0(t,x,0,0)\geq 0.$$

From the definition of F we see that W is a subsolution of (4.2). Similarly, we can prove that W is a viscosity supersolution of (4.2). Therefore, W is a viscosity solution of (4.2).

□

Case 2. We suppose that σ depends on z, and does not depend on u.

Now equation (3.1) becomes the following one

$$\begin{cases} dX_s^{t,x;u}=b(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)ds+\sigma(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u})dB_s,\\ dY_s^{t,x;u}=-f(s,X_s^{t,x;u},Y_s^{t,x;u},Z_s^{t,x;u},u_s)ds+Z_s^{t,x;u}dB_s,\qquad s\in[t,T],\\ X_t^{t,x;u}=x,\ Y_T^{t,x;u}=\Phi(X_T^{t,x;u}).\end{cases}\tag{4.22}$$

The related HJB equation is the following PDE combined with the algebraic equation:

$$\begin{cases} \frac{\partial}{\partial t}W(t,x)+H(t,x,W(t,x),V(t,x))=0,\\ V(t,x)=DW(t,x).\sigma(t,x,W(t,x),V(t,x)),\qquad (t,x)\in[0,T)\times\mathbb{R}^n,\\ W(T,x)=\Phi(x),\qquad x\in\mathbb{R}^n.\end{cases}\tag{4.23}$$

In this case

$$\begin{aligned} H(t,x,W(t,x),V(t,x))=\sup_{u\in U}\{&DW(t,x).b(t,x,W(t,x),V(t,x),u)\\ &+\tfrac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(t,x,W(t,x),V(t,x))D^2W(t,x))+f(t,x,W(t,x),V(t,x),u)\},\end{aligned}$$

where $t\in[0,T], x\in\mathbb{R}^n$.

We also give the definition of viscosity solution for this kind of PDE.

Definition 4.2. *A real-valued continuous function $W \in C([0,T]\times\mathbb{R}^n)$ is called*
(i) *a viscosity subsolution of equation (4.23) if $W(T,x) \le \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R}^n)$ satisfying the monotonicity condition (B2)'-(ii) and for all $(t,x) \in [0,T)\times\mathbb{R}^n$ such that $W-\varphi$ attains a local maximum at (t,x),*

$$\begin{cases} \frac{\partial\varphi}{\partial t}(t,x) + H(t,x,\varphi(t,x),\psi(t,x)) \ge 0, \\ \textit{where } \psi \textit{ is the unique solution of the following algebraic equation:} \\ \psi(t,x) = D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x)). \end{cases}$$

(ii) *a viscosity supersolution of equation (4.23) if $W(T,x) \ge \Phi(x)$, for all $x \in \mathbb{R}^n$, and if for all functions $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R}^n)$ satisfying the monotonicity condition (B2)'-(ii) and for all $(t,x) \in [0,T)\times\mathbb{R}^n$ such that $W-\varphi$ attains a local minimum at (t,x),*

$$\begin{cases} \frac{\partial\varphi}{\partial t}(t,x) + H(t,x,\varphi(t,x),\psi(t,x)) \le 0, \\ \textit{where } \psi \textit{ is the unique solution of the following algebraic equation:} \\ \psi(t,x) = D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x)). \end{cases}$$

(iii) *a viscosity solution of equation (4.23) if it is both a viscosity sub- and supersolution of equation (4.23).*

Remark 4.3. *In this case we need the following technical assumption:*
(B6) $\beta_2 > 0$;
(B7) $G\sigma(s,x,y,z)$ is continuous in s, uniformly with respect to $(x,y,z) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$.

Theorem 4.2. *Under the assumptions (B2), (B4), (B5), (B6) and (B7), the value function W is a viscosity solution of (4.23).*

Proof. Obviously, $W(T,x) = \Phi(x)$, $x \in \mathbb{R}^n$. We prove only that W is a viscosity subsolution, that it is also a viscosity supersolution can be proved similarly. We suppose that $\varphi \in C^3_{l,b}([0,T]\times\mathbb{R}^n)$ satisfying the monotonicity condition (B2)'-(ii) and that $(t,x) \in [0,T)\times\mathbb{R}^n$ is such that $W-\varphi$ attains its maximum at (t,x). Since W is continuous and of at most linear growth, we can replace the condition of a local maximum by that of a global one in the definition of the viscosity subsolution. Without loss of generality we may assume that $\varphi(t,x) = W(t,x)$. We consider the following equation:

$$\begin{cases} d\overline{X}^u_s = b(s,\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s,u_s)ds + \sigma(s,\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s)dB_s, \\ d\overline{Y}^u_s = -f(s,\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s,u_s)ds + \overline{Z}^u_s dB_s, \quad s \in [t,t+\delta], \\ \overline{X}^u_t = x, \ \overline{Y}^u_{t+\delta} = \varphi(t+\delta,\overline{X}^u_{t+\delta}), \quad 0 \le \delta \le T-t. \end{cases} \tag{4.24}$$

From Proposition 6.4 and Proposition 6.5 in Appendix, we know there exists sufficiently small $0 < \bar\delta_1 < T-t$ such that for any $0 \le \delta \le \bar\delta_1$, FBSDE (4.24) has a unique solution $(\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s)_{s\in[t,t+\delta]} \in \mathcal{S}^2(t,t+\delta;\mathbb{R}^n)\times\mathcal{S}^2(t,t+\delta;\mathbb{R})\times\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$, and for $p \ge 2$,

$$\begin{array}{ll} \text{(i)} & E[\sup\limits_{t\le s\le t+\delta}|\overline{X}^u_s|^p + \sup\limits_{t\le s\le t+\delta}|\overline{Y}^u_s|^p + (\int_t^{t+\delta}|\overline{Z}^u_s|^2ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \le C(1+|x|^p), \quad \text{P-a.s.}; \\ \text{(ii)} & E[\sup\limits_{t\le s\le t+\delta}|\overline{X}^u_s - x|^p \mid \mathcal{F}_t] \le C\delta^{\frac{p}{2}}(1+|x|^p), \quad \text{P-a.s.}; \\ \text{(iii)} & E[(\int_t^{t+\delta}|\overline{Z}^u_s|^2ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \le C\delta^{\frac{p}{2}}(1+|x|^p), \quad \text{P-a.s.} \end{array} \tag{4.25}$$

According to the definition of the backward stochastic semigroup for fully coupled FBSDE, we have

$$G^{t,x;u}_{s,t+\delta}[\varphi(t+\delta,\overline{X}^u_{t+\delta})] = \overline{Y}^u_s, \quad s \in [t,t+\delta].$$

And due to the DPP (Theorem 3.1), we have

$$\varphi(t,x) = W(t,x) = \operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}[W(t+\delta,\widetilde{X}^{t,x;u}_{t+\delta})], \ 0 \le \delta \le \bar\delta_1,$$

where $\widetilde{X}^{t,x;u}$ is defined by FBSDE (3.10).

From $\varphi(s,y) \geq W(s,y),\ (s,y) \in [0,T) \times \mathbb{R}^n$, and the monotonicity property of $G^{t,x;u}_{t,t+\delta}[\cdot]$ (see Theorem 6.2) we obtain

$$\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}} G^{t,x;u}_{t,t+\delta}[\varphi(t+\delta,\overline{X}^u_{t+\delta})] - \varphi(t,x) \geq 0,\ 0 \leq \delta \leq \bar{\delta}_1. \tag{4.26}$$

Now we set

$$\begin{array}{rcl} \overline{Y}^{1,u}_s & = & \overline{Y}^u_s - \varphi(s,\overline{X}^u_s) \\ & = & \int_s^{t+\delta} f(r,\overline{X}^u_r,\overline{Y}^u_r,\overline{Z}^u_r,u_r)dr - \int_s^{t+\delta} \overline{Z}^u_r dB_r + \varphi(t+\delta,\overline{X}^u_{t+\delta}) - \varphi(s,\overline{X}^u_s). \end{array} \tag{4.27}$$

Applying Itô's formula to $\varphi(s,\overline{X}^u_s)$, and setting $\overline{Z}^{1,u}_s = \overline{Z}^u_s - D\varphi(s,\overline{X}^u_s).\sigma(s,\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s)$, we obtain

$$\left\{\begin{array}{rcl} \overline{Y}^{1,u}_s & = & \int_s^{t+\delta}[\frac{\partial}{\partial r}\varphi(r,\overline{X}^u_r) + \frac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(r,\overline{X}^u_r,\overline{Y}^u_r,\overline{Z}^u_r)D^2\varphi(r,\overline{X}^u_r)) \\ & & +D\varphi(r,\overline{X}^u_r).b(r,\overline{X}^u_r,\overline{Y}^u_r,\overline{Z}^u_r,u_r) + f(r,\overline{X}^u_r,\overline{Y}^u_r,\overline{Z}^u_r,u_r)]dr - \int_s^{t+\delta}\overline{Z}^{1,u}_r dB_r, \\ \overline{Z}^{1,u}_s & = & \overline{Z}^u_s - D\varphi(s,\overline{X}^u_s).\sigma(s,\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s). \end{array}\right. \tag{4.28}$$

From (4.26) and (4.27), we have

$$\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}} \overline{Y}^{1,u}_t \geq 0. \tag{4.29}$$

Define

$$\begin{array}{rcl} L(s,x,y,z,u) & = & \frac{\partial}{\partial s}\varphi(s,x) + D\varphi(s,x).b(s,x,y+\varphi(s,x),z,u) + \frac{1}{2}\mathrm{tr}\,(\sigma\sigma^T(s,x,y+\varphi(s,x),z)D^2\varphi(s,x)) \\ & & +f(s,x,y+\varphi(s,x),z,u),\ (s,x,y,z,u) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U. \end{array}$$

Notice now that

$$\begin{array}{ll} \text{(i)} & |L(s,x,y,z,u) - L(s,x,y',z',u)| \leq C(1+|x|+|y|+|y'|+|z|+|z'|)(|y-y'|+|z-z'|); \\ \text{(ii)} & |L(s,x,y,z,u)| \leq C(1+|x|^2+|y|^2+|z|^2),\quad \forall (s,x,y,z,u) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d\times U. \end{array}$$

Therefore, equation (4.28) can be written into the following form:

$$\left\{\begin{array}{rcl} d\overline{Y}^{1,u}_s & = & -L(s,\overline{X}^u_s,\overline{Y}^{1,u}_s,\overline{Z}^u_s,u_s)ds + \overline{Z}^{1,u}_s dB_s,\quad s\in[t,t+\delta], \\ \overline{Z}^u_s & = & \overline{Z}^{1,u}_s + D\varphi(s,\overline{X}^u_s).\sigma(s,\overline{X}^u_s,\overline{Y}^{1,u}_s + \varphi(s,\overline{X}^u_s),\overline{Z}^u_s),\quad s\in[t,t+\delta], \\ \overline{Y}^{1,u}_{t+\delta} & = & 0, \end{array}\right. \tag{4.30}$$

Obviously, (4.30) has a solution $(\overline{Y}^{1,u}_s,\overline{Z}^{1,u}_s) \in \mathcal{S}^2(t,t+\bar{\delta}_1;\mathbb{R})\times\mathcal{H}^2(t,t+\bar{\delta}_1;\mathbb{R}^d)$, because (4.24) has a unique solution $(\overline{X}^u_s,\overline{Y}^u_s,\overline{Z}^u_s)_{s\in[t,t+\delta]}$, and $\overline{Y}^{1,u}_s = \overline{Y}^u_s - \varphi(s,\overline{X}^u_s)$, $\overline{Z}^{1,u}_s = \overline{Z}^u_s - D\varphi(s,\overline{X}^u_s).\sigma(s,\overline{X}^u_s,\overline{Y}^{1,u}_s + \varphi(s,\overline{X}^u_s),\overline{Z}^u_s)$ solves (4.30).

In order to complete the proof of Theorem 4.2, we need the following lemmas.

We need to consider the following BSDE combined with an algebraic equation:

$$\left\{\begin{array}{rcl} d\overline{Y}^{2,u}_s & = & -L(s,x,0,\hat{Z}^u_s,u_s)ds + \overline{Z}^{2,u}_s dB_s,\quad s\in[t,t+\delta], \\ \hat{Z}^u_s & = & \overline{Z}^{1,u}_s + D\varphi(s,x).\sigma(s,x,\overline{Y}^{1,u}_s + \varphi(s,x),\hat{Z}^u_s),\quad s\in[t,t+\delta], \\ \overline{Y}^{2,u}_{t+\delta} & = & 0, \end{array}\right. \tag{4.31}$$

where $u(\cdot) \in \mathcal{U}_{t,t+\delta}$. For it, we have to study first the algebraic equation.

Remark 4.4. *For $m=1$, the matrix G becomes a vector in $\mathbb{R}^n$, and without loss of generality, we may assume $G=(1,0,\cdots,0)\in\mathbb{R}^n$. Thus, we have the following conditions from the monotonicity condition (B2):*

$$\begin{array}{ll} \text{(i)} & \langle\sigma_1(t,x,y,z)-\sigma_1(t,x,y,\bar{z}),z-\bar{z}\rangle \leq -\beta_2|z-\bar{z}|^2; \\ \text{(ii)} & G^T D\varphi(s,x) \geq 0,\ i.e., D_{x_1}\varphi(s,x)\geq 0,\ D_{x_i}\varphi(s,x)=0,\ 2\leq i\leq n. \end{array} \tag{4.32}$$

Indeed, (i) follows from (B2)-(i), and (ii) follows from (B2)'-(ii) satisfied by φ : $\langle\varphi(s,x)-\varphi(s,\bar{x}),G(x-\bar{x})\rangle \geq 0,\ \varphi\in C^3_{l,b}([0,T]\times\mathbb{R}^n)$.

We have the following important Representation Theorem for the solution of the algebraic equation.

Proposition 4.1. *For any $s \in [0,T]$, $\zeta \in \mathbb{R}^d$, $y \in \mathbb{R}$, $\bar{x} \in \mathbb{R}^n$, there exists a unique z such that $z = \zeta + D\varphi(s,\bar{x}).\sigma(s,\bar{x},y+\varphi(s,\bar{x}),z)$. That means, the solution z can be written as $z = h(s,\bar{x},y,\zeta)$, where the function h is Lipschitz with respect to y,ζ, and $|h(s,\bar{x},y,\zeta)| \le C(1+|\bar{x}|+|y|+|\zeta|)$. The constant C is independent of $s,\bar{x},y,\zeta$. And $z = h(s,\bar{x},y,\zeta)$ is continuous with respect to s.*

Proof. *First Step.* From Remark 4.4, we can prove that the equation $z = D\varphi(s,\bar{x}).\sigma(s,\bar{x},\varphi(s,\bar{x}),z)$ has a unique solution z.
Indeed, by fixing $(s,\bar{x})$ and setting $a := D_{x_1}\varphi(s,\bar{x}),\ \sigma_1(z) = \sigma_1(s,\bar{x},\varphi(s,\bar{x}),z)$, we only need to consider the equation $z = a\sigma_1(z)$.

(1) If $a = 0$, then $z = 0$ is the solution.

(2) Let $a > 0$, and $\zeta \in \mathbb{R}^d$. We choose a small $\delta \in (0,1)$ to make the mapping $z \mapsto a\delta\sigma_1(z)$ Lipschitz with Lipschitz constant $L_\delta < 1$, and we set $\delta_0 = \frac{1}{1+[\frac{1}{\delta}]}$, where $[z]$ represents the integer part of the real nonnegative number z. Then the mapping $z \mapsto a\delta_0\sigma_1(z)$ is still Lipschitz with Lipschitz constant $L_{\delta_0} < 1$. For simplicity of the notations we still use δ to denote δ_0. Obviously, there exists a unique fixed point z_δ such that $z_\delta = \zeta + \delta a\sigma_1(z_\delta)$. We now consider $z_\delta^{n+1} = (\zeta + \delta a\sigma_1(z_\delta^n)) + \delta a\sigma_1(z_\delta^{n+1}),\ n \ge 1,\ z_\delta^1 = 0$, and we put $\bar{z}_\delta^n := z_\delta^n - z_\delta^{n-1},\ n > 1$. Then

$$
\begin{aligned}
|\bar{z}_\delta^{n+1}|^2 &= \delta a\langle \sigma_1(z_\delta^n) - \sigma_1(z_\delta^{n-1}), \bar{z}_\delta^{n+1}\rangle + \delta a\langle \sigma_1(z_\delta^{n+1}) - \sigma_1(z_\delta^n), \bar{z}_\delta^{n+1}\rangle \\
&\le L_\delta|\bar{z}_\delta^n|\cdot|\bar{z}_\delta^{n+1}| - \beta_2\delta a|\bar{z}_\delta^{n+1}|^2.
\end{aligned}
$$

Thus, putting $\bar{\beta}_2 := a\delta\beta_2 > 0$, we have $(1+\bar{\beta}_2)|\bar{z}_\delta^{n+1}|^2 \le |\bar{z}_\delta^n|\cdot|\bar{z}_\delta^{n+1}| \le \frac{1}{2}|\bar{z}_\delta^n|^2 + \frac{1}{2}|\bar{z}_\delta^{n+1}|^2$, from where we get $(\frac{1}{2}+\bar{\beta}_2)|\bar{z}_\delta^{n+1}|^2 \le \frac{1}{2}|\bar{z}_\delta^n|^2,\ n \ge 1$. Therefore, there exists a unique $z_\delta^{'}$, such that $z_\delta^{'} = \zeta + 2\delta a\sigma_1(z_\delta^{'})$.
Let $N \ge 1$ such that $N\delta = 1$ and $1 \le k \le N$. Suppose that $\zeta \in \mathbb{R}^d$, there exists a unique z_δ such that $z_\delta = \zeta + k\delta a\sigma_1(z_\delta)$. Now we consider the equation $z_\delta^{n+1} = (\zeta + \delta a\sigma_1(z_\delta^n)) + k\delta a\sigma_1(z_\delta^{n+1}),\ n \ge 1,\ z_\delta^1 = 0$. Using the above argument we see that also the equation $z_\delta = \zeta + (k+1)\delta a\sigma_1(z_\delta)$ has a unique fixed point z_δ, for all $\zeta \in \mathbb{R}^d$. This completes the proof of the first step.

Second step. From the above, since for any $\zeta \in \mathbb{R}^d$, there exists a unique z of the equation $z = \zeta + D\varphi(s,\bar{x}).\sigma(s,\bar{x},y+\varphi(s,\bar{x}),z)(= \zeta + D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},y+\varphi(s,\bar{x}),z))$. z is uniquely determined by $(s,\bar{x},y,\zeta)$, and we can put $z = h(s,\bar{x},y,\zeta)$. This function h is measurable and it is Lipschitz with respect to $y,\ \zeta$.

Indeed, for any $\bar{y},\ \bar{\zeta},\ \hat{y},\ \hat{\zeta}$, we consider:

$$\bar{z} = \bar{\zeta} + D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\bar{y}+\varphi(s,\bar{x}),\bar{z}),\ \ \hat{z} = \hat{\zeta} + D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\hat{y}+\varphi(s,\bar{x}),\hat{z}).$$

Then, taking into account Remark 4.4-(i) and $\varphi \in C_{l,b}^3([0,T]\times\mathbb{R}^n)$,

$$
\begin{aligned}
&\langle \bar{z}-\hat{z}, \bar{z}-\hat{z}\rangle \\
=\ &\langle \bar{\zeta}-\hat{\zeta} + D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\bar{y}+\varphi(s,\bar{x}),\bar{z}) - D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\hat{y}+\varphi(s,\bar{x}),\hat{z}), \bar{z}-\hat{z}\rangle \\
=\ &\langle \bar{\zeta}-\hat{\zeta}, \bar{z}-\hat{z}\rangle + \langle D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\bar{y}+\varphi(s,\bar{x}),\bar{z}) - D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\bar{y}+\varphi(s,\bar{x}),\hat{z}), \bar{z}-\hat{z}\rangle \\
&+\langle D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\bar{y}+\varphi(s,\bar{x}),\hat{z}) - D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},\hat{y}+\varphi(s,\bar{x}),\hat{z}), \bar{z}-\hat{z}\rangle \\
\le\ &C|\bar{\zeta}-\hat{\zeta}|^2 + \tfrac{1}{8}|\bar{z}-\hat{z}|^2 + C(-\beta_2)|\bar{z}-\hat{z}|^2 + C|\bar{y}-\hat{y}||\bar{z}-\hat{z}| \\
\le\ &C|\bar{\zeta}-\hat{\zeta}|^2 + \tfrac{1}{4}|\bar{z}-\hat{z}|^2 + C|\bar{y}-\hat{y}|^2.
\end{aligned}
$$

Therefore, we have $|\bar{z}-\hat{z}| \le C(|\bar{\zeta}-\hat{\zeta}| + |\bar{y}-\hat{y}|)$.

Similarly we can prove $|h(s,\bar{x},y,\zeta)| \le C(1+|\bar{x}|+|y|+|\zeta|)$, where the constant C is independent of $s,\ \bar{x},\ y,\ \zeta$. Indeed, we have

$$
\begin{aligned}
&\langle z, z\rangle = \langle \zeta + D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},y+\varphi(s,\bar{x}),z), z\rangle \\
=\ &\langle \zeta, z\rangle + \langle D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},y+\varphi(s,\bar{x}),z) - D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},y+\varphi(s,\bar{x}),0), z\rangle \\
&+\langle D_{x_1}\varphi(s,\bar{x})\sigma_1(s,\bar{x},y+\varphi(s,\bar{x}),0), z\rangle \\
\le\ &C|\zeta|^2 + \tfrac{1}{4}|z|^2 + C(-\beta_2)|z|^2 + C(1+|\bar{x}|+|y|)^2,
\end{aligned}
$$

which implies $|z| \le C(1+|\bar{x}|+|y|+|\zeta|)$.

The fact that $z = h(s, \bar{x}, y, \zeta)$ is continuous with respect to s can be proved similarly. Indeed, let $z_1 = \zeta + D_{x_1}\varphi(s_1, x)\sigma_1(s_1, x, y + \varphi(s_1, x), z_1)$, $z_2 = \zeta + D_{x_1}\varphi(s_2, x)\sigma_1(s_2, x, y + \varphi(s_2, x), z_2)$, then,

$$\begin{array}{rl} & \langle z_1 - z_2, z_1 - z_2\rangle \\ = & \langle D_{x_1}\varphi(s_1, x)\sigma_1(s_1, x, y + \varphi(s_1, x), z_1) - D_{x_1}\varphi(s_1, x)\sigma_1(s_1, x, y + \varphi(s_1, x), z_2), z_1 - z_2\rangle \\ & + \langle D_{x_1}\varphi(s_1, x)\sigma_1(s_1, x, y + \varphi(s_1, x), z_2) - D_{x_1}\varphi(s_2, x)\sigma_1(s_2, x, y + \varphi(s_2, x), z_2), z_1 - z_2\rangle \\ \leq & \langle D_{x_1}\varphi(s_1, x)\sigma_1(s_1, x, y + \varphi(s_1, x), z_2) - D_{x_1}\varphi(s_2, x)\sigma_1(s_2, x, y + \varphi(s_2, x), z_2), z_1 - z_2\rangle \\ \leq & C|s_1 - s_2|(1 + |x| + |y|)|z_1 - z_2| + C(|\sigma_1(s_2, x, y + \varphi(s_1, x), z_2) - \sigma_1(s_1, x, y + \varphi(s_1, x), z_2)| \\ & + |s_1 - s_2|)|z_1 - z_2| \\ \leq & C|s_1 - s_2|^2(1 + |x|^2 + |y|^2) + C(|\sigma_1(s_2, x, y + \varphi(s_1, x), z_2) - \sigma_1(s_1, x, y + \varphi(s_1, x), z_2)|^2 \\ & + |s_1 - s_2|^2) + \frac{1}{2}|z_1 - z_2|^2, \end{array}$$

therefore, we have

$$\begin{array}{rcl} |z_1 - z_2|^2 & \leq & C|s_1 - s_2|^2(1 + |x|^2 + |y|^2) + C|\sigma_1(s_2, x, y + \varphi(s_1, x), z_2) - \sigma_1(s_1, x, y + \varphi(s_1, x), z_2)|^2 \\ & \leq & C|s_1 - s_2|^2(1 + |x|^2 + |y|^2) + C|\rho(\delta)|^2, \end{array}$$

where $\rho(\delta) := \sup_{x \in \mathbb{R}^n,\ y \in \mathbb{R},\ z \in \mathbb{R}^d} |\sigma_1(s_2, x, y, z) - \sigma_1(s_1, x, y, z)|$, $s_1, s_2 \in [0, T]$, for $|s_1 - s_2| \leq \delta$, then from (B7) we have $\rho(\delta) \to 0$, as $\delta \to 0$. It follows that $z = h(s, \bar{x}, y, \zeta)$ is continuous with respect to s. $\square$

Lemma 4.5. *For every $u \in \mathcal{U}_{t,t+\delta}$,*

$$E[\int_t^{t+\delta} (|\overline{Y}_s^{1,u}| + |\overline{Z}_s^{1,u}|)ds \mid \mathcal{F}_t] \leq C\delta^{\frac{5}{4}}, \quad P\text{-a.s.},\ 0 \leq \delta \leq \bar{\delta}_1, \tag{4.33}$$

where the constant C is independent of the control u and of $\delta > 0$.

Proof. From equation (4.30), we have that $\overline{Z}_s^u$ can be written as $\overline{Z}_s^u = h(s, \overline{X}_s^u, \overline{Y}_s^{1,u}, \overline{Z}_s^{1,u})$, where h satisfies the properties given in Proposition 4.1. Let $F(s, x, y, z, u) = L(s, x, y, h(s, x, y, z), u)$. Then, (4.30) can be rewritten as follows

$$\left\{ \begin{array}{rcl} d\overline{Y}_s^{1,u} & = & -F(s, \overline{X}_s^u, \overline{Y}_s^{1,u}, \overline{Z}_s^{1,u}, u_s)ds + \overline{Z}_s^{1,u} dB_s, \quad s \in [t, t+\delta], \\ \overline{Y}_{t+\delta}^{1,u} & = & 0. \end{array} \right. \tag{4.34}$$

Then the proof is similar to the proof of Lemma 4.1, we can get

$$|\overline{Y}_s^{1,u}| \leq C\delta^{\frac{1}{2}}(1 + |\overline{X}_s^u|), \quad \text{P-a.s.}, \quad s \in [t, t+\delta]. \tag{4.35}$$

On the other hand,

$$|\overline{Z}_s^{1,u}| = |\overline{Z}_s^u - D\varphi(s, \overline{X}_s^u).\sigma(s, \overline{X}_s^u, \overline{Y}_s^u, \overline{Z}_s^u)| \leq C(1 + |\overline{X}_s^u| + |\overline{Y}_s^u| + |\overline{Z}_s^u|), \quad \text{P-a.s.} \tag{4.36}$$

And, from (4.27) and (4.35) we know,

$$|\overline{Y}_s^u| \leq C(1 + |\overline{X}_s^u|), \quad \text{P-a.s.},\ s \in [t, t+\delta]. \tag{4.37}$$

From (4.34), (4.35), (4.36), (4.37) and (4.25),

$$\begin{array}{rl} & |\overline{Y}_t^{1,u}|^2 + E[\int_t^{t+\delta} |\overline{Z}_r^{1,u}|^2 dr \mid \mathcal{F}_t] = 2E[\int_t^{t+\delta} \overline{Y}_r^{1,u} F(r, \overline{X}_r^u, \overline{Y}_r^{1,u}, \overline{Z}_r^{1,u}, u_r)dr \mid \mathcal{F}_t] \\ \leq & CE[\int_t^{t+\delta} |\overline{Y}_r^{1,u}|(1 + |\overline{X}_r^u|^2 + |\overline{Y}_r^{1,u}|^2 + |\overline{Z}_r^{1,u}|^2)dr \mid \mathcal{F}_t] \\ \leq & C\delta^{\frac{1}{2}} E[\int_t^{t+\delta}(1 + |\overline{X}_r^u|^2 + |\overline{X}_r^u|^3)dr \mid \mathcal{F}_t] + C\delta^{\frac{1}{2}} E[\int_t^{t+\delta}(1 + |\overline{X}_r^u|)|\overline{Z}_r^u|^2 dr \mid \mathcal{F}_t] \leq C\delta^{\frac{3}{2}}, \quad \text{P-a.s.} \end{array} \tag{4.38}$$

Therefore,

$$\begin{array}{rl} & E[\int_t^{t+\delta}(|\overline{Y}_s^{1,u}| + |\overline{Z}_s^{1,u}|)ds \mid \mathcal{F}_t] \leq C\delta^{\frac{1}{2}} E[\int_t^{t+\delta}(1 + |\overline{X}_r^u|)dr \mid \mathcal{F}_t] + C\delta^{\frac{1}{2}}(E[\int_t^{t+\delta} |\overline{Z}_r^{1,u}|^2 dr \mid \mathcal{F}_t])^{\frac{1}{2}} \\ \leq & C\delta^{\frac{5}{4}}, \quad \text{P-a.s.},\ 0 \leq \delta < \bar{\delta}_1. \end{array}$$

$\square$

Remark 4.5. *From (4.36), (4.37) and (4.25) we get*

$$E[(\int_t^{t+\delta}|\overline{Z}_s^{1,u}|^2ds)^2 \mid \mathcal{F}_t] \le C\delta^2, \quad P\text{-a.s.} \tag{4.39}$$

Remark 4.6. *From Proposition 4.1 and the fact that the solution* $(\overline{Y}^{1,u}, \overline{Z}^{1,u})$ *belongs to* $\mathcal{S}^2(t,t+\delta;\mathbb{R}) \times \mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$*, we see that the unique solution* $\hat{Z}_s^u$ *of the equation*

$$\hat{Z}_s^u = \overline{Z}_s^{1,u} + D\varphi(s,x).\sigma(s,x,\overline{Y}_s^{1,u} + \varphi(s,x), \hat{Z}_s^u), \ \ s \in [t,t+\delta],$$

belongs to $\mathcal{H}^2(t,t+\delta;\mathbb{R}^d)$ *and* $\hat{Z}_s^u = h(s,x,\overline{Y}_s^{1,u}, \overline{Z}_s^{1,u})$*. Similar to Remark 4.5, we know*
(i) $E[\int_t^{t+\delta}|\hat{Z}_s^u|^2ds \mid \mathcal{F}_t] \le C\delta,$ *P-a.s.*; (ii) $E[(\int_t^{t+\delta}|\hat{Z}_s^u|^2ds)^2 \mid \mathcal{F}_t] \le C\delta^2,$ *P-a.s.*
Then, from Lemma 2.1 in [16] BSDE (4.31) has a unique solution $(\overline{Y}^{2,u}, \overline{Z}^{2,u})$.

Lemma 4.6. *For every* $u \in \mathcal{U}_{t,t+\delta}$*, we have*

$$|\overline{Y}_t^{1,u} - \overline{Y}_t^{2,u}| \le C\delta^{\frac{5}{4}}, \quad P\text{-a.s.}, \quad 0 \le \delta \le \bar{\delta}_1,$$

where C *is independent of the control process* u *and of* $\delta > 0$.

Proof. Similar to the proof of Lemma 4.2 we set $g(s) = L(s,\overline{X}_s^u,0,\overline{Z}_s^u,u_s) - L(s,x,0,\overline{Z}_s^u,u_s),\ \rho_0(r) = (1+|x|^2+|\overline{Z}_s^u|^2)(r+r^2),\ r \ge 0$. Obviously, $|g(s)| \le C\rho_0(|\overline{X}_s^u - x|)$, for $s \in [t,t+\delta],\ (t,x) \in [0,T)\times\mathbb{R}^n,\ u \in \mathcal{U}_{t,t+\delta}$. Therefore, we have, from equations (4.30) and (4.31), estimates (4.25), (4.35) and (4.39),

$$\begin{array}{rl} & |\overline{Y}_t^{1,u} - \overline{Y}_t^{2,u}| = |E[(\overline{Y}_t^{1,u} - \overline{Y}_t^{2,u}) \mid \mathcal{F}_t]| \\ = & |E[\int_t^{t+\delta}(L(s,\overline{X}_s^u,\overline{Y}_s^{1,u},\overline{Z}_s^u,u_s) - L(s,x,0,\hat{Z}_s^u,u_s))ds \mid \mathcal{F}_t]| \\ \le & CE[\int_t^{t+\delta}(\rho_0(|\overline{X}_s^u - x|) + C(1+|\overline{X}_s^u| + |\overline{Y}_s^{1,u}| + |\overline{Z}_s^u|)|\overline{Y}_s^{1,u}| \\ & +C(1+|x|+|\overline{Z}_s^u| + |\overline{Z}_s^u - \hat{Z}_s^u|)|\overline{Z}_s^u - \hat{Z}_s^u|)ds \mid \mathcal{F}_t] \\ \le & C\delta^{\frac{5}{4}} + CE[\int_t^{t+\delta}|\overline{Z}_s^u - \hat{Z}_s^u|ds \mid \mathcal{F}_t] + CE[\int_t^{t+\delta}|\overline{Z}_s^u||\overline{Z}_s^u - \hat{Z}_s^u|ds \mid \mathcal{F}_t] \\ & +CE[\int_t^{t+\delta}|\overline{Z}_s^u - \hat{Z}_s^u|^2ds \mid \mathcal{F}_t]. \end{array} \tag{4.40}$$

Furthermore,

$$\begin{array}{rl} & \langle \hat{Z}_s^u - \overline{Z}_s^u, \hat{Z}_s^u - \overline{Z}_s^u\rangle \\ = & \langle D_{x_1}\varphi(s,x)\sigma_1(s,x,\overline{Y}_s^{1,u}+\varphi(s,x),\hat{Z}_s^u) - D_{x_1}\varphi(s,x)\sigma_1(s,x,\overline{Y}_s^{1,u}+\varphi(s,x),\overline{Z}_s^u), \hat{Z}_s^u - \overline{Z}_s^u\rangle \\ & +\langle D_{x_1}\varphi(s,x)\sigma_1(s,x,\overline{Y}_s^{1,u}+\varphi(s,x),\overline{Z}_s^u) - D_{x_1}\varphi(s,x)\sigma_1(s,\overline{X}_s^u,\overline{Y}_s^{1,u}+\varphi(s,\overline{X}_s^u),\overline{Z}_s^u), \hat{Z}_s^u - \overline{Z}_s^u\rangle \\ & +\langle D_{x_1}\varphi(s,x)\sigma_1(s,\overline{X}_s^u,\overline{Y}_s^{1,u}+\varphi(s,\overline{X}_s^u),\overline{Z}_s^u) - D_{x_1}\varphi(s,\overline{X}_s^u)\sigma_1(s,\overline{X}_s^u,\overline{Y}_s^{1,u}+\varphi(s,\overline{X}_s^u),\overline{Z}_s^u), \hat{Z}_s^u - \overline{Z}_s^u\rangle \\ \le & -\beta_2|\hat{Z}_s^u - \overline{Z}_s^u|^2 + C|\overline{X}_s^u - x||\hat{Z}_s^u - \overline{Z}_s^u| + C|\overline{X}_s^u - x|(1+|\overline{X}_s^u| + |\overline{Y}_s^{1,u}| + |\overline{Z}_s^u|)|\hat{Z}_s^u - \overline{Z}_s^u| \\ \le & C|\overline{X}_s^u - x|^2 + \frac{1}{2}|\hat{Z}_s^u - \overline{Z}_s^u|^2 + C|\overline{X}_s^u - x|^2(1+|\overline{X}_s^u| + |\overline{Y}_s^{1,u}| + |\overline{Z}_s^u|)^2, \end{array}$$

from Proposition 4.1 and Remark 4.4. Therefore, we have

$$|\overline{Z}_s^u - \hat{Z}_s^u| \le C(1+|\overline{X}_s^u|)|\overline{X}_s^u - x| + C|\overline{X}_s^u - x|(|\overline{Y}_s^{1,u}| + |\overline{Z}_s^u|).$$

Then, from Lemma 4.5 and (4.25), the proof is complete. □

We now consider the following equation

$$\left\{\begin{array}{rcl} d\overline{Y}_s^{3,u} & = & -L(s,x,0,\psi(s,x),u_s)ds + \overline{Z}_s^{3,u}dB_s, \ \ s \in [t,t+\delta], \\ \psi(s,x) & = & D\varphi(s,x).\sigma(s,x,\varphi(s,x),\psi(s,x)), \ \ s \in [t,t+\delta], \\ \overline{Y}_{t+\delta}^{3,u} & = & 0. \end{array}\right. \tag{4.41}$$

Lemma 4.7. *For every* $u \in \mathcal{U}_{t,t+\delta}$*, we have*

$$|\overline{Y}_t^{2,u} - \overline{Y}_t^{3,u}| \le C\delta^{\frac{5}{4}}, \ \ P\text{-a.s.}, \quad 0 \le \delta \le \bar{\delta}_1,$$

where C *is independent of the control process* u *and of* $\delta > 0$.

Proof. From (4.31) and (4.41), we get

$$\begin{array}{rl} & |\overline{Y}_t^{2,u}-\overline{Y}_t^{3,u}| = |E[\int_t^{t+\delta}(L(s,x,0,\hat{Z}_s^u,u_s)-L(s,x,0,\psi(s,x),u_s))ds \mid \mathcal{F}_t]| \\ \leq & CE[\int_t^{t+\delta}(1+|x|+|\hat{Z}_s^u|)|\hat{Z}_s^u-\psi(s,x)|ds \mid \mathcal{F}_t]. \end{array}$$

From Remark 4.6, $\hat{Z}_s^u = h(s,x,\overline{Y}_s^{1,u},\overline{Z}_s^{1,u})$, and from Proposition 4.1, $\psi(s,x)=h(s,x,0,0)$, hence we obtain $|\hat{Z}_s^u-\psi(s,x)|\leq C(|\overline{Y}_s^{1,u}|+|\overline{Z}_s^{1,u}|)$. Notice

$$\begin{array}{rl} & E[\int_t^{t+\delta}|\hat{Z}_s^u||\hat{Z}_s^u-\psi(s,x)|ds \mid \mathcal{F}_t] \\ \leq & C(E[\int_t^{t+\delta}|\hat{Z}_s^u|^2ds \mid \mathcal{F}_t])^{\frac{1}{2}}(E[\int_t^{t+\delta}(|\overline{Y}_s^{1,u}|+|\overline{Z}_s^{1,u}|)^2ds \mid \mathcal{F}_t])^{\frac{1}{2}} \leq C\delta^{\frac{5}{4}},\ \text{ P-a.s.,} \end{array}$$

where the last inequality is due to (4.38) and (i) in Remark 4.6. Furthermore, from Lemma 4.5, we have $|\overline{Y}_t^{2,u}-\overline{Y}_t^{3,u}|\leq C\delta^{\frac{5}{4}}$, P-a.s. □

Lemma 4.8. *Let $\overline{Y}_0(\cdot)$ be the solution of the following ordinary differential equation combined with an algebraic equation:*

$$\left\{\begin{array}{lcl} d\overline{Y}_0(s) & = & -L_0(s,x,0,\psi(s,x))ds,\ \ s\in[t,t+\delta], \\ \psi(s,x) & = & D\varphi(s,x).\sigma(s,x,\varphi(s,x),\psi(s,x)),\ \ s\in[t,t+\delta], \\ \overline{Y}_0(t+\delta) & = & 0, \end{array}\right. \tag{4.42}$$

where the function L_0 is defined by

$$L_0(s,x,0,z)=\sup_{u\in U}L(s,x,0,z,u),\ \ (s,x,z)\in[t,t+\delta]\times\mathbb{R}^n\times\mathbb{R}^d.$$

Then, P-a.s.,

$$\overline{Y}_0(t)=\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}\overline{Y}_t^{3,u}.$$

Proof. Since $L_0(s,x,0,z)=\sup\limits_{u\in U}L(s,x,0,z,u)$, we have

$$L_0(s,x,0,\psi(s,x))\geq L(s,x,0,\psi(s,x),u_s),\ \ s\in[t,t+\delta],\ \text{ for all } u\in\mathcal{U}_{t,t+\delta},$$

we have $\overline{Y}_0(t)\geq\overline{Y}_t^{3,u}$, P-a.s., for any $u\in\mathcal{U}_{t,t+\delta}$. Indeed, (4.42) can be regarded as a BSDE with the solution $(Y_s,Z_s)=(\overline{Y}_0(s),0)$. This allows to apply the comparison theorem of BSDEs.

On the other hand, since $L_0(s,x,0,z)=\sup\limits_{u\in U}L(s,x,0,z,u)$, there exists a measurable function $\tilde{u}:[t,t+\delta]\times\mathbb{R}^n\times\mathbb{R}^d\to U$, such that $L_0(s,x,0,\psi(s,x))=L(s,x,0,\psi(s,x),\tilde{u}(s,x,\psi(s,x)))$, for all $s\in[t,t+\delta]$. We put $\tilde{u}_s=\tilde{u}(s,x,\psi(s,x))$, $s\in[t,t+\delta]$, obviously $\tilde{u}\in\mathcal{U}_{t,t+\delta}$, and $L_0(s,x,0,\psi(s,x))=L(s,x,0,\psi(s,x),\tilde{u}_s)$, $s\in[t,t+\delta]$. Consequently, from the uniqueness of the solution of the BSDE, we have $(\overline{Y}_t^{3,\tilde{u}},\overline{Z}_t^{3,\tilde{u}})=(\overline{Y}_0(t),0)$, and particularly, $\overline{Y}_0(t)=\overline{Y}_t^{3,\tilde{u}}$, P-a.s. Therefore, $\overline{Y}_0(t)=\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}\overline{Y}_t^{3,u}$. □

We are now able to finish the proof of Theorem 4.2.

Indeed, from (4.29) we know that

$$\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}\overline{Y}_t^{1,u}\geq 0,\ \text{ P-a.s.}$$

Thus, from the Lemmas 4.6 and 4.7 we get $\operatorname{ess\,sup}_{u\in\mathcal{U}_{t,t+\delta}}\overline{Y}_t^{3,u}\geq -C\delta^{\frac{5}{4}}$, P-a.s. Thus, by Lemma 4.8, $\overline{Y}_0(t)\geq -C\delta^{\frac{5}{4}}$, P-a.s., where $\overline{Y}_0$ is the unique solution of (4.42). Consequently,

$$-C\delta^{\frac{1}{4}}\leq\frac{1}{\delta}\overline{Y}_0(t)=\frac{1}{\delta}\int_t^{t+\delta}L_0(s,x,0,\psi(s,x))ds,\ \ \delta>0,$$

from where, thanks to the continuity of $s\mapsto L_0(s,x,0,\psi(s,x))$, it follows that

$$\sup_{u\in U}L(t,x,0,\psi(t,x),u)=L_0(t,x,0,\psi(t,x))\geq 0,$$

where $\psi(t,x)=D\varphi(t,x).\sigma(t,x,\varphi(t,x),\psi(t,x))$ and from the definition of L we see that W is a viscosity subsolution of (4.23). Similarly, we can prove that W is a viscosity supersolution of (4.23). Therefore, W is a viscosity solution of (4.23).

□

5 Examples

Now we give two examples associated with the two cases studied above. For simplification, we set $m = n = d = 1$, and $G = 1$. In the first example, σ does not depend on z, but depends on u.

Example 5.1. *We consider the following fully coupled FBSDE:*

$$\begin{cases} dX_s^{t,x;u} = (3X_s^{t,x;u} + 5Z_s^{t,x;u})ds + (4X_s^{t,x;u} - 5Y_s^{t,x;u} + u_s)dB_s, \\ dY_s^{t,x;u} = -(2X_s^{t,x;u} + 3Y_s^{t,x;u} + 4Z_s^{t,x;u} + u_s)ds + Z_s^{t,x;u}dB_s, \quad s \in [t,T], \\ X_t^{t,x;u} = x, \ Y_T^{t,x;u} = X_T^{t,x;u}, \end{cases} \tag{5.1}$$

where $u \in \mathcal{U}$ is an admissible control.

For a given admissible control u, the coefficients of equation (5.1) satisfy the assumptions (B1), (B2) and (B4), then there exists a unique solution $(X^{t,x;u}, Y^{t,x;u}, Z^{t,x;u})$. We define

$$W(t,x) = \operatorname{ess\,sup}_{u \in \mathcal{U}_{t,T}} Y_t^{t,x;u}, \tag{5.2}$$

it follows from Theorem 4.1 that $W(t,x)$ is the viscosity solution of the following PDE:

$$\begin{cases} \frac{\partial}{\partial t}W(t,x) + \sup\limits_{u\in U}\{\frac{1}{2}\frac{\partial^2}{\partial x^2}W(t,x)(4x - 5W(t,x) + u)^2 + \frac{\partial}{\partial x}W(t,x)(3x + 5\frac{\partial}{\partial x}W(t,x)(4x - 5W(t,x) + u)) \\ \qquad\qquad +2x + 3W(t,x) + 4\frac{\partial}{\partial x}W(t,x)(4x - 5W(t,x) + u)\} = 0, \quad (t,x) \in [0,T) \times \mathbb{R}, \\ W(T,x) = x. \end{cases}$$

In the following example, σ depends on z, but does not depend on u.

Example 5.2. *We consider the following fully coupled FBSDE:*

$$\begin{cases} dX_s^{t,x;u} = (-(X_s^{t,x;u})^+ - 4Y_s^{t,x;u} + u_s)ds + (-X_s^{t,x;u} - L_\sigma Z_s^{t,x;u})dB_s, \\ dY_s^{t,x;u} = -(2X_s^{t,x;u} - (Y_s^{t,x;u})^+ - Z_s^{t,x;u} + u_s)ds + Z_s^{t,x;u}dB_s, \quad s \in [t,T], \\ X_t^{t,x;u} = x, \ Y_T^{t,x;u} = X_T^{t,x;u}, \end{cases} \tag{5.3}$$

where the constant $L_\sigma > 0$ is sufficiently small, $u \in \mathcal{U}$ is an admissible control.

It is easy to check that the coefficients of equation (5.3) satisfy the assumptions (B1), (B2), (B4), (B5), (B6) and (B7), hence there exists a unique solution $(X^{t,x;u}, Y^{t,x;u}, Z^{t,x;u})$. We define

$$W(t,x) = \operatorname{ess\,sup}_{u \in \mathcal{U}_{t,T}} Y_t^{t,x;u}, \tag{5.4}$$

and we associate (5.3) with the following partial differential equation,

$$\begin{cases} \frac{\partial}{\partial t}W(t,x) + \sup\limits_{u\in U}\{\frac{1}{2}\frac{\partial^2}{\partial x^2}W(t,x)(x + L_\sigma V(t,x))^2 + \frac{\partial}{\partial x}W(t,x)(-4W(t,x) - x^+ + u) + 2x - W^+(t,x) \\ \qquad\qquad -V(t,x) + u\} = 0, \\ V(t,x) = \frac{\partial}{\partial x}W(t,x)(-x - L_\sigma V(t,x)), \quad (t,x) \in [0,T) \times \mathbb{R}, \\ W(T,x) = x. \end{cases} \tag{5.5}$$

Therefore, from Theorem 4.2, $W(t,x)$ defined by (5.4) is the viscosity solution of (5.5).

6 Appendix

In this subsection we prove some basic important estimates for fully coupled FBSDEs under monotonic assumptions, and present new estimates and new generalized comparison theorem for FBSDEs on small time interval. Let us now give four mappings:

$b : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n$, $\qquad \sigma : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{n\times d}$,
$f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $\qquad \Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}$,

$(b(t,x,y,z))_{t\in[0,T]}$, $(\sigma(t,x,y,z))_{t\in[0,T]}$, $(f(t,x,y,z))_{t\in[0,T]}$ are $\mathbb{F}$-progressively measurable for each $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$, and $\Phi(x)$ is $\mathcal{F}_T$-measurable for each $x\in\mathbb{R}^n$, which satisfy (B1) and (B2), and also

(C1) there exists a constant $K\geq 0$ such that, for any $t\in[0,T]$, $(x,y,z),\ (x',y',z')\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$, P-a.s.,
$|l(t,x,y,z)-l(t,x',y',z')|\leq K(|x-x'|+|y-y'|+|z-z'|)$, where $l=b,\ \sigma,\ f,$ respectively, and
$|\Phi(x)-\Phi(x')|\leq K(|x-x'|)$;

(C2) there exists a constant $L\geq 0$ such that, for any $t\in[0,T]$, $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$, P-a.s.,
$|b(t,x,y,z)|+|\sigma(t,x,y,z)|+|f(t,x,y,z)|+|\Phi(x)|\leq L(1+|x|+|y|+|z|)$.

We consider the following FBSDE parameterized by the initial condition $(t,\zeta)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$:

$$\left\{\begin{array}{l} dX_s^{t,\zeta}=b(s,X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})ds+\sigma(s,X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})dB_s,\\ dY_s^{t,\zeta}=-f(s,X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})ds+Z_s^{t,\zeta}dB_s,\qquad s\in[t,T],\\ X_t^{t,\zeta}=\zeta,\ Y_T^{t,\zeta}=\Phi(X_T^{t,\zeta}).\end{array}\right. \tag{6.1}$$

Proposition 6.1. *Under the assumptions (B1), (B2), (C1) and (C2), for any $0\leq t\leq T$ and the associated initial states $\zeta,\ \zeta'\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, we have the following estimates, P-a.s.:*

(i) $E[\sup\limits_{t\leq s\leq T}|X_s^{t,\zeta}-X_s^{t,\zeta'}|^2+\sup\limits_{t\leq s\leq T}|Y_s^{t,\zeta}-Y_s^{t,\zeta'}|^2+\int_t^T|Z_s^{t,\zeta}-Z_s^{t,\zeta'}|^2ds\mid\mathcal{F}_t]\leq C|\zeta-\zeta'|^2$;

(ii) $E[\sup\limits_{t\leq s\leq T}|X_s^{t,\zeta}|^2+\sup\limits_{t\leq s\leq T}|Y_s^{t,\zeta}|^2+\int_t^T|Z_s^{t,\zeta}|^2ds\mid\mathcal{F}_t]\leq C(1+|\zeta|^2)$.

If σ also satisfies:

(C3) *for any $t\in[0,T]$, for any $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^d$, P-a.s., $|\sigma(t,x,y,z)|\leq L(1+|x|+|y|)$,*

then we can get

(iii) $E[\sup\limits_{t\leq s\leq t+\delta}|X_s^{t,\zeta}-\zeta|^2\mid\mathcal{F}_t]\leq C\delta(1+|\zeta|^2)$, *P-a.s.,* $0\leq\delta\leq T-t$.

Proof. From Lemma 2.1 we know, for the initial states $\zeta,\ \zeta'\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, FBSDE (6.1) has a unique solution $(X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})_{s\in[t,T]}\in\mathcal{S}^2(t,T;\mathbb{R}^n)\times\mathcal{S}^2(t,T;\mathbb{R})\times\mathcal{H}^2(t,T;\mathbb{R}^d)$, and $(X_s^{t,\zeta'},Y_s^{t,\zeta'},Z_s^{t,\zeta'})_{s\in[t,T]}\in\mathcal{S}^2(t,T;\mathbb{R}^n)\times\mathcal{S}^2(t,T;\mathbb{R})\times\mathcal{H}^2(t,T;\mathbb{R}^d)$, respectively. We define $\hat{X}_s=X_s^{t,\zeta}-X_s^{t,\zeta'}$, $\hat{Y}_s=Y_s^{t,\zeta}-Y_s^{t,\zeta'}$, $\hat{Z}_s=Z_s^{t,\zeta}-Z_s^{t,\zeta'}$, $\Delta h(s)=h(s,X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})-h(s,X_s^{t,\zeta'},Y_s^{t,\zeta'},Z_s^{t,\zeta'})$, where $h=b,\ \sigma,\ f,\ A$, respectively. Applying Itô's formula to $|\hat{X}_s|^2$ we have

$$\begin{array}{rcl} E[|\hat{X}_s|^2\mid\mathcal{F}_t] &=& |\zeta-\zeta'|^2+E[\int_t^s(2\hat{X}_r\Delta b(r)+|\Delta\sigma(r)|^2)dr\mid\mathcal{F}_t]\\ &\leq& |\zeta-\zeta'|^2+CE[\int_t^s(|\hat{X}_r|^2+|\hat{Y}_r|^2+|\hat{Z}_r|^2)dr\mid\mathcal{F}_t],\ t\leq s\leq T.\end{array}$$

Then, from the Gronwall inequality, we obtain

$$E[|\hat{X}_s|^2\mid\mathcal{F}_t]\leq C(|\zeta-\zeta'|^2+E[\int_t^s(|\hat{Y}_r|^2+|\hat{Z}_r|^2)dr\mid\mathcal{F}_t]),\ \text{P-a.s.},\ t\leq s\leq T. \tag{6.2}$$

Apply Itô's formula to $e^{\beta s}|\hat{Y}_s|^2$, taking β large enough, using standard methods for BSDEs we can get

$$\begin{array}{rl} & E[|\hat{Y}_s|^2\mid\mathcal{F}_t]+E[\int_s^T|\hat{Y}_r|^2dr\mid\mathcal{F}_t]+E[\int_s^T|\hat{Z}_r|^2dr\mid\mathcal{F}_t]\\ \leq & C(E[|\hat{X}_T|^2\mid\mathcal{F}_t]+E[\int_t^T|\hat{X}_r|^2dr\mid\mathcal{F}_t]),\ t\leq s\leq T.\end{array} \tag{6.3}$$

Then from (6.2) it follows that

$$\begin{array}{rl} & E[|\hat{Y}_s|^2\mid\mathcal{F}_t]+E[\int_s^T|\hat{Y}_r|^2dr\mid\mathcal{F}_t]+E[\int_s^T|\hat{Z}_r|^2dr\mid\mathcal{F}_t]\\ \leq & C|\zeta-\zeta'|^2+CE[\int_t^T(|\hat{X}_r|^2+|\hat{Y}_r|^2+|\hat{Z}_r|^2)dr\mid\mathcal{F}_t],\ t\leq s\leq T.\end{array} \tag{6.4}$$

On the other hand, applying Itô's formula to $\langle G\hat{X}_r,\hat{Y}_r\rangle$, from the assumption (B2) we get

$$\begin{array}{rl} & \langle G\hat{X}_s,\hat{Y}_s\rangle=E[\langle G\hat{X}_T,\hat{Y}_T\rangle\mid\mathcal{F}_s]-E[\int_s^T\langle\Delta A(r),(\hat{X}_r,\hat{Y}_r,\hat{Z}_r)\rangle dr\mid\mathcal{F}_s]\\ \geq & E[\mu_1|G\hat{X}_T|^2\mid\mathcal{F}_s]+E[\beta_1\int_s^T|G\hat{X}_r|^2dr\mid\mathcal{F}_s]+E[\int_s^T\beta_2(|G^T\hat{Y}_r|^2+|G^T\hat{Z}_r|^2)dr\mid\mathcal{F}_s].\end{array} \tag{6.5}$$

Therefore, $\langle G\hat{X}_s, \hat{Y}_s\rangle \geq 0,\ t \leq s \leq T,$ P-a.s.
If $\beta_2 > 0$, then we get

$$\begin{array}{rcl}\langle G\hat{X}_t, \hat{Y}_t\rangle & = & E[\langle G\hat{X}_s, \hat{Y}_s\rangle \mid \mathcal{F}_t] - E[\int_t^s \langle \Delta A(r), (\hat{X}_r, \hat{Y}_r, \hat{Z}_r)\rangle dr \mid \mathcal{F}_t] \\ & \geq & \beta_2 E[\int_t^s (|G^T\hat{Y}_r|^2 + |G^T\hat{Z}_r|^2)dr \mid \mathcal{F}_t],\ t \leq s \leq T,\ \text{P-a.s.}\end{array} \tag{6.6}$$

Therefore,

$$E[\int_t^s (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq C\langle G\hat{X}_t, \hat{Y}_t\rangle,\ t \leq s \leq T,\ \text{P-a.s.} \tag{6.7}$$

Then, from (6.2) we can get

$$E[|\hat{X}_s|^2 \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2 + C\langle G\hat{X}_t, \hat{Y}_t\rangle,\ t \leq s \leq T,\ \text{P-a.s.} \tag{6.8}$$

From (6.4) we have

$$E[|\hat{Y}_s|^2 \mid \mathcal{F}_t] + E[\int_s^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2 + C\langle G\hat{X}_t, \hat{Y}_t\rangle,\ t \leq s \leq T,\ \text{P-a.s.} \tag{6.9}$$

Therefore, recalling that $\hat{X}_t = X_t^{t,\zeta} - X_t^{t,\zeta'} = \zeta - \zeta'$,

$$|\hat{Y}_t|^2 \leq C|\zeta - \zeta'|^2 + C\langle G\hat{X}_t, \hat{Y}_t\rangle \leq C|\zeta - \zeta'|^2 + C|\hat{X}_t||\hat{Y}_t| \leq C|\zeta - \zeta'|^2 + C|\hat{X}_t|^2 + \tfrac{1}{2}|\hat{Y}_t|^2,\ \text{P-a.s.}$$

which means $|\hat{Y}_t| \leq C|\zeta - \zeta'|,$ P-a.s. Furthermore, from (6.8), (6.9), we can get

$$E[|\hat{X}_s|^2 \mid \mathcal{F}_t] + E[|\hat{Y}_s|^2 \mid \mathcal{F}_t] + E[\int_s^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2,\ t \leq s \leq T,\ \text{P-a.s.}$$

If $\beta_2 = 0$, then according to assumption (B2), we have $\beta_1 > 0,\ \mu_1 > 0,\ m = n = 1$, i.e., $G \in \mathbb{R}$. From (6.5), $E[|\hat{X}_T|^2 \mid \mathcal{F}_t] + E[\int_t^T |\hat{X}_r|^2 dr \mid \mathcal{F}_t] \leq CG\hat{X}_t \cdot \hat{Y}_t,\ C > 0$. From (6.3), $|\hat{Y}_t|^2 + E[\int_t^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq CG\hat{X}_t \cdot \hat{Y}_t \leq C|\zeta - \zeta'|^2 + \frac{1}{2}|\hat{Y}_t|^2$, therefore, $|\hat{Y}_t|^2 + E[\int_t^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2$. Furthermore, from (6.2), $E[|\hat{X}_s|^2 \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2,\ t \leq s \leq T,$ P-a.s. Then from (6.3) we get $E[|\hat{Y}_s|^2 \mid \mathcal{F}_t] + E[\int_s^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2,\ t \leq s \leq T,$ P-a.s. From above we always have

$$E[|\hat{X}_s|^2 \mid \mathcal{F}_t] + E[|\hat{Y}_s|^2 \mid \mathcal{F}_t] + E[\int_s^T (|\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \leq C|\zeta - \zeta'|^2,\ t \leq s \leq T,\ \text{P-a.s.} \tag{6.10}$$

Finally, from equation (6.1) and Buckholder-Davis-Gundy inequality we have

$$\begin{array}{rcl}E[\sup\limits_{t\leq s\leq T} |\hat{X}_s|^2 \mid \mathcal{F}_t] & \leq & 3|\zeta - \zeta'|^2 + CE[\int_t^T |\Delta b(r)|^2 dr \mid \mathcal{F}_t] + CE[\int_t^T |\Delta\sigma(r)|^2 dr \mid \mathcal{F}_t] \\ & \leq & 3|\zeta - \zeta'|^2 + CE[\int_t^T (|\hat{X}_r|^2 + |\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \\ & \leq & C|\zeta - \zeta'|^2,\ \text{P-a.s.};\end{array}$$

and

$$\begin{array}{rcl}E[\sup\limits_{t\leq s\leq T} |\hat{Y}_s|^2 \mid \mathcal{F}_t] & \leq & CE[|X_T^{t,\zeta} - X_T^{t,\zeta'}|^2 \mid \mathcal{F}_t] + CE[\int_t^T (|\hat{X}_r|^2 + |\hat{Y}_r|^2 + |\hat{Z}_r|^2)dr \mid \mathcal{F}_t] \\ & \leq & C|\zeta - \zeta'|^2,\ \text{P-a.s.}\end{array}$$

Similarly we can prove (ii) by making full use of the monotonic assumption (B2).
Now it is not hard to prove (iii). Indeed, from (C3) we obtain,

$$\begin{array}{rl} & E[\sup\limits_{t\leq s\leq t+\delta} |X_s^{t,\zeta} - \zeta|^2 \mid \mathcal{F}_t] \\ \leq & 2E[|\int_t^{t+\delta} b(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})dr|^2 \mid \mathcal{F}_t] + CE[\int_t^{t+\delta} |\sigma(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})|^2 dr \mid \mathcal{F}_t] \\ \leq & C\delta E[\int_t^{t+\delta} (1 + |X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2 + |Z_r^{t,\zeta}|^2)dr \mid \mathcal{F}_t] + CE[\int_t^{t+\delta} (1 + |X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2)dr \mid \mathcal{F}_t] \\ \leq & C\delta E[\sup\limits_{t\leq r\leq t+\delta} (|X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2) + \int_t^{t+\delta} |Z_r^{t,\zeta}|^2 dr \mid \mathcal{F}_t] + C\delta + C\delta E[\sup\limits_{t\leq r\leq t+\delta} (|X_r^{t,\zeta}|^2 + |Y_r^{t,\zeta}|^2) \mid \mathcal{F}_t] \\ \leq & C\delta(1 + |\zeta|^2),\ \text{P-a.s.}\end{array}$$

$\square$

Remark 6.1. *From Proposition 6.1, we have the following inequalities:*

$$|Y_t^{t,\zeta}| \leq C(1+|\zeta|); \qquad |Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta-\zeta'|, \quad P\text{-a.s.}, \tag{6.11}$$

where the constant $C>0$ *depends only on the Lipschitz constants and linear growth constants of* $b,\ \sigma,\ f$ *and* Φ.

Remark 6.2. *Let* $\Phi(x)=\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R})$. *From the proof of Proposition 6.1 we see that: under the assumptions (B1), (B2)-(i), (C1) and (C2), the statements of Proposition 6.1-(i) and (ii) still hold true; if furthermore, assumption (C3) holds, then we also have the same result as Proposition 6.1-(iii).*

Now we consider the continuous dependence of the fully coupled FBSDE on the terminal condition.

Proposition 6.2. *Suppose the assumptions (B1), (B2), (C1) and (C2) are satisfied. For any* $0\leq t\leq T$, *the associated initial state* $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$ *and* $\xi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R})$, *we let* $(X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})_{s\in[t,T]}$ *be the solution of FBSDE (6.1) associated with* (b,σ,f,ζ,Φ), *and* $(\overline{X}_s^{t,\zeta},\overline{Y}_s^{t,\zeta},\overline{Z}_s^{t,\zeta})_{s\in[t,T]}$ *be the solution of FBSDE (6.1) associated with* $(b,\sigma,f,\zeta,\Phi+\xi)$. *Then we have*

$$|Y_t^{t,\zeta}-\overline{Y}_t^{t,\zeta}|^2+E[\int_t^T|Y_r^{t,\zeta}-\overline{Y}_r^{t,\zeta}|^2dr\mid\mathcal{F}_t]+E[\int_t^T|Z_r^{t,\zeta}-\overline{Z}_r^{t,\zeta}|^2dr\mid\mathcal{F}_t]\leq CE[\xi^2\mid\mathcal{F}_t],\ P\text{-a.s.}$$

The proof is similar to Proposition 6.1, we omit it here.

Let us now introduce the random field:

$$u(t,x)=Y_s^{t,x}\mid_{s=t},\quad (t,x)\ \in\ [0,T]\times\mathbb{R}^n,$$

where $Y^{t,x}$ is the solution of FBSDE (6.1) with the initial state $x\in\mathbb{R}^n$.

As a consequence of Remark 6.1 we have that, for all $t\in[0,T]$, P-a.s.,

$$\begin{array}{l}\text{(i) } |u(t,x)-u(t,y)|\leq C|x-y|,\ \text{for all } x,y\in\mathbb{R}^n;\\ \text{(ii) } |u(t,x)|\leq C(1+|x|),\ \text{for all } x\in\mathbb{R}^n.\end{array} \tag{6.12}$$

Remark 6.3. *Moreover, it is well known that, under the additional assumption that the functions*

$$b,\sigma,f\ \textit{and}\ \Phi\ \textit{are deterministic}, \tag{6.13}$$

u *is also a deterministic function of* (t,x).

The random field u and $Y^{t,\zeta},(t,\zeta)\in[0,T]\times L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, are related by the following theorem.

Theorem 6.1. *Under the assumptions (B1) and (B2), for any* $t\in[0,T]$ *and* $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, *we have*

$$u(t,\zeta)=Y_t^{t,\zeta},\quad P\text{-a.s.}$$

The proof of Theorem 6.1 is similar to that in Peng [16] for the decoupled FBSDE, or we can also refer to the proof of Theorem A.2 in [2].

Remark 6.4. *From Theorem 6.1, we can obtain* $Y_s^{t,\zeta}=Y_s^{s,X_s^{t,\zeta}}=u(s,X_s^{t,\zeta})$.

Proposition 6.3. *Under the assumptions (B1), (B2), (C1), (C2) and (C3), for any* $p\geq2$, $0\leq t\leq T$ *and the associated initial state* $\zeta\in L^p(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, *there exists* $\tilde{\delta}_0>0$, *which depends on* p *and Lipschitz constant* K *and the linear growth constant* L, *such that*

$$\begin{array}{ll}\text{(i)} & E[\sup\limits_{t\leq s\leq t+\tilde{\delta}_0}|X_s^{t,\zeta}|^p+\sup\limits_{t\leq s\leq t+\tilde{\delta}_0}|Y_s^{t,\zeta}|^p+(\int_t^{t+\tilde{\delta}_0}|Z_s^{t,\zeta}|^2ds)^{\frac{p}{2}}\mid\mathcal{F}_t]\leq C_p(1+|\zeta|^p),\ P\text{-a.s.};\\ \text{(ii)} & E[\sup\limits_{t\leq s\leq t+\delta}|X_s^{t,\zeta}-\zeta|^p\mid\mathcal{F}_t]\leq C_p\delta^{\frac{p}{2}}(1+|\zeta|^p),\ P\text{-a.s.},\ 0\leq\delta\leq\tilde{\delta}_0,\end{array}$$

where $(X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})_{s\in[t,T]}$ *is the solution of FBSDE (6.1) associated with* (b,σ,f,ζ,Φ).

Proof. From Remarks 6.1 and 6.4, we have

$$|Y_s^{t,\zeta}| = |Y_s^{s,X_s^{t,\zeta}}| \leq C(1 + |X_s^{t,\zeta}|), \text{ P-a.s.} \tag{6.14}$$

Since $Y_t^{t,\zeta} = Y_s^{t,\zeta} + \int_t^s f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})dr - \int_t^s Z_r^{t,\zeta}dB_r,\ t \leq s \leq t+\delta$, from Buckholder-Davis-Gundy inequality,

$$\begin{aligned}
&E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C_p E[\sup_{t\leq s\leq t+\delta} |\int_t^s Z_r^{t,\zeta}dB_r|^p \mid \mathcal{F}_t] \\
\leq\ & C_p E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p + (\int_t^{t+\delta} f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})ds)^p \mid \mathcal{F}_t] \\
\leq\ & C_p E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t] + C_p E[(\int_t^{t+\delta}(1 + |X_s^{t,\zeta}| + |Y_s^{t,\zeta}| + |Z_s^{t,\zeta}|)ds)^p \mid \mathcal{F}_t] \\
\leq\ & C_p E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t] + C_p\delta^p + C_p\delta^p E[\sup_{t\leq s\leq t+\delta} |X_s^{t,\zeta}|^p \mid \mathcal{F}_t] + C_p\delta^p E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t] \\
& + C_p\delta^{\frac{p}{2}} E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
=\ & C_p\delta^p + C_p\delta^p E[\sup_{t\leq s\leq t+\delta} |X_s^{t,\zeta}|^p \mid \mathcal{F}_t] + (C_p + C_p\delta^p) E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t] \\
& + C_p\delta^{\frac{p}{2}} E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t],
\end{aligned}$$

there exists $\delta_0 > 0$, such that $1 - C_p\delta_0^{\frac{p}{2}} > 0$, then we get, for any $0 \leq \delta \leq \delta_0$, P-a.s.,

$$E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C_p\delta^p + C_p\delta^p E[\sup_{t\leq s\leq t+\delta} |X_s^{t,\zeta}|^p \mid \mathcal{F}_t] + (C_p + C_p\delta^p) E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t]. \tag{6.15}$$

On the other hand, for $t \leq s \leq T$, from (6.1) and (6.14),

$$\begin{aligned}
&E[\sup_{t\leq r\leq s} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t] \\
\leq\ & C_p E[(\int_t^s |b(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})|dr)^p \mid \mathcal{F}_t] + C_p E[(\int_t^s |\sigma(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
\leq\ & C_p E[(\int_t^s (1 + |X_r^{t,\zeta} - \zeta| + |\zeta| + |Z_r^{t,\zeta}|)dr)^p \mid \mathcal{F}_t] + C_p E[(\int_t^s (1 + |X_r^{t,\zeta}|)^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
\leq\ & C_p(1 + |\zeta|^p)(s-t)^{\frac{p}{2}} + C_p(s-t)^{\frac{p}{2}} E[(\int_t^s |Z_r^{t,\zeta}|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t] + C_p E[\int_t^s |X_r^{t,\zeta} - \zeta|^p dr \mid \mathcal{F}_t],
\end{aligned}$$

from Gronwall inequality,

$$E[\sup_{t\leq r\leq s} |X_r^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t] \leq C_p(1 + |\zeta|^p)(s-t)^{\frac{p}{2}} + C_p(s-t)^{\frac{p}{2}} E[(\int_t^s |Z_r^{t,\zeta}|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t], \text{ P-a.s., } t \leq s \leq T. \tag{6.16}$$

Then, from (6.15), (6.14) and (6.16) we have

$$\begin{aligned}
&E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \\
\leq\ & C_p\delta^p(1 + |\zeta|^p) + C_p|\zeta|^p + C_p + (C_p + C_p\delta^p) E[\sup_{t\leq s\leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t] \\
\leq\ & C_p + C_p|\zeta|^p + C_p\delta^{\frac{p}{2}}(1 + |\zeta|^p) + (C_p + C_p\delta^p) C_p\delta^{\frac{p}{2}} E[(\int_t^s |Z_r^{t,\zeta}|^2 dr)^{\frac{p}{2}} \mid \mathcal{F}_t],
\end{aligned}$$

taking $0 < \widetilde{\delta}_0 \leq \delta_0$, such that $1 - (C_p + C_p\widetilde{\delta}_0^p)C_p\widetilde{\delta}_0^{\frac{p}{2}} > 0$, then $0 \leq \delta \leq \widetilde{\delta}_0$,

$$E[(\int_t^{t+\delta} |Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq C_p(1 + |\zeta|^p), \text{ P-a.s.}$$

From (6.16), we get $E[\sup_{t\leq s\leq t+\delta} |X_s^{t,\zeta} - \zeta|^p \mid \mathcal{F}_t] \leq C_p\delta^{\frac{p}{2}}(1 + |\zeta|^p),\ \text{P-a.s.},\ 0 \leq \delta \leq \widetilde{\delta}_0$.

From (6.14) we have $E[\sup_{t\leq s\leq t+\delta} |Y_s^{t,\zeta}|^p \mid \mathcal{F}_t] \leq C_p(1 + |\zeta|^p),\ \text{P-a.s.},\ 0 \leq \delta \leq \widetilde{\delta}_0$. □

Proposition 6.4. *We suppose the assumptions (C1), (C2) and (C4) hold true, where the assumption (C4) is the following hypothesis:*
(C4) the Lipschitz constant $L_\sigma \geq 0$ of σ with respect to z is sufficiently small, i.e., there exists small enough $L_\sigma \geq 0$ such that, for all $t \in [0,T],\ u \in U,\ x_1, x_2 \in \mathbb{R}^n,\ y_1, y_2 \in \mathbb{R},\ z_1, z_2 \in \mathbb{R}^d$, P-a.s.,

$$|\sigma(t, x_1, y_1, z_1, u) - \sigma(t, x_2, y_2, z_2, u)| \leq K(|x_1 - x_2| + |y_1 - y_2|) + L_\sigma|z_1 - z_2|.$$

Then, there exists a constant $0 < \delta_0$, only depending on the Lipschitz constant K, such that for every $0 \le \delta \le \delta_0$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, FBSDE (6.1) has a unique solution $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})_{s\in[t,t+\delta]}$ on the time interval $[t,\ t+\delta]$.

Proof. Let us give any $0 < T \le T_0$, and observe that for any pair $v = (y, z) \in \mathcal{H}^2(t, T; \mathbb{R}^{1+d})$ there exists a unique solution $V = (Y, Z) \in \mathcal{H}^2(t, T; \mathbb{R}^{1+d})$ to the following decoupled FBSDE:

$$\left\{\begin{array}{l} dX_s = b(s, X_s, y_s, z_s)ds + \sigma(s, X_s, y_s, z_s)dB_s,\\ dY_s = -f(s, X_s, Y_s, Z_s)ds + Z_s dB_s, \qquad s \in [t, T],\\ X_t = \zeta,\ Y_T = \Phi(X_T). \end{array}\right. \tag{6.17}$$

We are going to prove that there exists a constant $0 < \delta_0$, only depending on the Lipschitz constant K, such that for every $0 \le \delta \le \delta_0$ the mapping defined by

$$I(v) = V : \mathcal{H}^2(t, t+\delta; \mathbb{R}^{1+d}) \to \mathcal{H}^2(t, t+\delta; \mathbb{R}^{1+d})$$

is a contraction.

Let $v^i = (y^i, z^i) \in \mathcal{H}^2(t, t+\delta; \mathbb{R}^{1+d})$, and $V^i = I(v^i),\ i = 1, 2$. We define $\widehat{v} = (y^1 - y^2, z^1 - z^2)$, and $\widehat{V} = (Y^1 - Y^2, Z^1 - Z^2),\ \widehat{X} = X^1 - X^2$. Then, we get

$$\begin{array}{rcl} E[\sup\limits_{t\le s\le r} |\widehat{X}_s|^2 \mid \mathcal{F}_t] & \le & 2E[(\int_t^r |b(s, X_s^1, y_s^1, z_s^1) - b(s, X_s^2, y_s^2, z_s^2)|ds)^2 \mid \mathcal{F}_t]\\ & & +8E[\int_t^r |\sigma(s, X_s^1, y_s^1, z_s^1) - \sigma(s, X_s^2, y_s^2, z_s^2)|^2 ds \mid \mathcal{F}_t]\\ & \le & 6(r-t)K^2 E[\int_t^r (|\widehat{X}_s|^2 + |\widehat{y}_s|^2 + |\widehat{z}_s|^2)ds \mid \mathcal{F}_t]\\ & & +24E[\int_t^r (K^2|\widehat{X}_s|^2 + K^2|\widehat{y}_s|^2 + L_\sigma^2|\widehat{z}_s|^2)ds \mid \mathcal{F}_t], \end{array} \tag{6.18}$$

and the Gronwall inequality yields

$$\begin{array}{l} E[\sup\limits_{t\le s\le T} |\widehat{X}_s|^2 \mid \mathcal{F}_t] \le CE[\int_t^T |\widehat{y}_s|^2 ds \mid \mathcal{F}_t] + (C(T-t) + CL_\sigma^2)E[\int_t^T |\widehat{z}_s|^2 ds \mid \mathcal{F}_t]\\ \le C(T-t)E[\sup\limits_{t\le s\le T} |\widehat{y}_s|^2 \mid \mathcal{F}_t] + (C(T-t) + CL_\sigma^2)E[\int_t^T |\widehat{z}_s|^2 ds \mid \mathcal{F}_t]. \end{array} \tag{6.19}$$

On the other hand, by using BSDE standard estimates combined with the help of (6.19), we get

$$\begin{array}{rl} & E[\sup\limits_{t\le s\le T} |\widehat{Y}_s|^2 + \int_t^T |\widehat{Z}_s|^2 ds]\\ \le & CE[|\Phi(X_T^1) - \Phi(X_T^2)|^2] + CE[\int_t^T |f(r, X_r^1, Y_r^1, Z_r^1) - f(r, X_r^2, Y_r^1, Z_r^1)|^2 dr]\\ \le & CE[|\widehat{X}_T|^2] + CE[\int_t^T |\widehat{X}_r|^2 dr]\\ \le & C(T-t)E[\sup\limits_{t\le s\le T} |\widehat{y}_s|^2] + (C(T-t) + CL_\sigma^2)E[\int_t^T |\widehat{z}_s|^2 ds]\\ \le & (C(T-t) + CL_\sigma^2)(E[\sup\limits_{t\le s\le T} |\widehat{y}_s|^2] + E[\int_t^T |\widehat{z}_s|^2 ds]). \end{array}$$

Notice L_σ is sufficiently small such that $CL_\sigma^2 \le \frac{1}{3}$. Then there exists $\delta_0 > 0$ such that $C\delta_0 + CL_\sigma^2 < \frac{1}{2}$, and therefore, for any $0 \le \delta \le \delta_0$, we have

$$E[\sup_{t\le s\le t+\delta} |\widehat{Y}_s|^2 + \int_t^{t+\delta} |\widehat{Z}_s|^2 ds] \le \tfrac{1}{2}(E[\sup_{t\le s\le t+\delta} |\widehat{y}_s|^2] + E[\int_t^{t+\delta} |\widehat{z}_s|^2 ds]). \tag{6.20}$$

It follows immediately that for any $0 \le \delta \le \delta_0$ this mapping I has a unique fixed point $I(V) = V$, i.e., FBSDE (6.1) has a unique solution $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})_{s\in[t,t+\delta]}$ on the time interval $[t,\ t+\delta]$. $\square$

Remark 6.5. *In fact, from the proof we see that $L_\sigma \ge 0$ with $CL_\sigma^2 < 1$ is sufficient for Proposition 6.4.*

Theorem 6.2. *(Generalized Comparison Theorem) We suppose the assumptions (C1), (C2) and (C4) are satisfied. Let $\delta_0 > 0$ be a constant, only depending on the Lipschitz constant K, such that for every $0 \le \delta \le \delta_0$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$, FBSDE (6.1) has a unique solution $(X_s^i, Y_s^i, Z_s^i)_{s\in[t,t+\delta]}$ associated with $(b, \sigma, f, \zeta, \Phi^i)$ on the time interval $[t,\ t+\delta]$, respectively. Then, if for any $0 \le \delta \le \delta_0$ we have $\Phi^1(X_{t+\delta}^2) \ge \Phi^2(X_{t+\delta}^2)$, P-a.s., (resp., $\Phi^1(X_{t+\delta}^1) \ge \Phi^2(X_{t+\delta}^1)$, P-a.s.), we also have $Y_t^1 \ge Y_t^2$, P-a.s.*

Proof. The proof is similar to that of Theorem 3.1 in Wu [18]. For notational simplification, we assume $d = n = 1$. We define $\widehat{X} = X^1 - X^2$, $\widehat{Y} = Y^1 - Y^2$, $\widehat{Z} = Z^1 - Z^2$, then $(\widehat{X}, \widehat{Y}, \widehat{Z})$ satisfies the following FBSDE:

$$\begin{array}{l} d\widehat{X}_s = (b_s^1\widehat{X}_s + b_s^2\widehat{Y}_s + b_s^3\widehat{Z}_s)ds + (\sigma_s^1\widehat{X}_s + \sigma_s^2\widehat{Y}_s + \sigma_s^3\widehat{Z}_s)dB_s, \\ d\widehat{Y}_s = -(f_s^1\widehat{X}_s + f_s^2\widehat{Y}_s + f_s^3\widehat{Z}_s)ds + \widehat{Z}_s dB_s, \\ \widehat{X}_t = 0,\ \widehat{Y}_{t+\delta} = \overline{\Phi}\widehat{X}_{t+\delta} + \Phi^1(X^2_{t+\delta}) - \Phi^2(X^2_{t+\delta}), \end{array} \tag{6.21}$$

where

$$\begin{array}{ll} l_s^1 = \frac{l(s,X_s^1,Y_s^1,Z_s^1) - l(s,X_s^2,Y_s^1,Z_s^1)}{X_s^1 - X_s^2}, & \widehat{X}_s \neq 0; \\ l_s^1 = 0, & \text{otherwise}; \\ l_s^2 = \frac{l(s,X_s^2,Y_s^1,Z_s^1) - l(s,X_s^2,Y_s^2,Z_s^1)}{Y_s^1 - Y_s^2}, & \widehat{Y}_s \neq 0; \\ l_s^2 = 0, & \text{otherwise}; \\ l_s^3 = \frac{l(s,X_s^2,Y_s^2,Z_s^1) - l(s,X_s^2,Y_s^2,Z_s^2)}{Z_s^1 - Z_s^2}, & \widehat{Z}_s \neq 0; \\ l_s^3 = 0, & \text{otherwise}, \end{array}$$

$l = b,\ \sigma,\ f$ respectively, and

$$\begin{array}{ll} \overline{\Phi} = \frac{\Phi^1(X^1_{t+\delta}) - \Phi^1(X^2_{t+\delta})}{X^1_{t+\delta} - X^2_{t+\delta}}, & \widehat{X}_{t+\delta} \neq 0; \\ \overline{\Phi} = 0, & \text{otherwise}. \end{array}$$

It's easy to check that (6.21) satisfies (C1), (C2) and (C4). Therefore, from Proposition 6.4 there exists a constant $0 < \delta_1 \le \delta_0$, such that for every $0 \le \delta \le \delta_1$, (6.21) has a unique solution on $[t, t+\delta]$, i.e., $(\widehat{X}, \widehat{Y}, \widehat{Z})$ is the unique solution of (6.21) on $[t, t+\delta]$, for every $0 \le \delta \le \delta_1$. Now we introduce the dual FBSDE

$$\begin{array}{l} dP_s = (f_s^2 P_s - b_s^2 Q_s - \sigma_s^2 K_s)ds + (f_s^3 P_s - b_s^3 Q_s - \sigma_s^3 K_s)dB_s, \\ dQ_s = (f_s^1 P_s - b_s^1 Q_s - \sigma_s^1 K_s)ds + K_s dB_s, \\ P_t = 1,\ Q_{t+\delta} = -\overline{\Phi}P_{t+\delta}. \end{array} \tag{6.22}$$

Similarly, (6.22) satisfies (C1), (C2) and (C4). Consequently, due to Proposition 6.4 there exists a constant $0 < \delta_2 \le \delta_1$, such that for every $0 \le \delta \le \delta_2$, (6.22) has a unique solution (P, Q, K) on $[t, t+\delta]$. Using Itô's formula to $\widehat{X}_s Q_s + \widehat{Y}_s P_s$, we deduce from the equations (6.21) and (6.22) that,

$$E[\widehat{X}_{t+\delta}(-\overline{\Phi}P_{t+\delta})|\mathcal{F}_t] + E[(\overline{\Phi}\widehat{X}_{t+\delta} + \Phi^1(X^2_{t+\delta}) - \Phi^2(X^2_{t+\delta}))P_{t+\delta}|\mathcal{F}_t] = \widehat{Y}_t,$$

i.e.,

$$\widehat{Y}_t = E[(\Phi^1(X^2_{t+\delta}) - \Phi^2(X^2_{t+\delta}))P_{t+\delta}|\mathcal{F}_t]. \tag{6.23}$$

Since $\Phi^1(X^2_{t+\delta}) \ge \Phi^2(X^2_{t+\delta})$, P-a.s., if we can prove $P_{t+\delta} \ge 0$, P-a.s., then we can get $\widehat{Y}_t \ge 0$, P-a.s.

For this we define the following stopping time: $\tau = \inf\{s > t : P_s = 0\} \wedge (t+\delta)$, and consider the following FBSDE (6.24) on $[\tau, t+\delta]$ (notice that $\tau > t$, since P is continuous and $P_t = 1$):

$$\begin{array}{l} d\widetilde{P}_s = (f_s^2 \widetilde{P}_s - b_s^2 \widetilde{Q}_s - \sigma_s^2 \widetilde{K}_s)ds + (f_s^3 \widetilde{P}_s - b_s^3 \widetilde{Q}_s - \sigma_s^3 \widetilde{K}_s)dB_s, \\ d\widetilde{Q}_s = (f_s^1 \widetilde{P}_s - b_s^1 \widetilde{Q}_s - \sigma_s^1 \widetilde{K}_s)ds + \widetilde{K}_s dB_s, \\ \widetilde{P}_\tau = 0,\ \widetilde{Q}_{t+\delta} = -\overline{\Phi}\widetilde{P}_{t+\delta}. \end{array} \tag{6.24}$$

Similarly to the equation (6.22) we see that, (6.24) satisfies (C1), (C2) and (C4), and therefore, from Proposition 6.4 there exists $0 < \delta_3 \le \delta_2$ such that for every $0 \le \delta \le \delta_3$, (6.24) has a unique solution $(\widetilde{P}, \widetilde{Q}, \widetilde{K})$ on $[\tau, t+\delta]$. Clearly, $(\widetilde{P}_s, \widetilde{Q}_s, \widetilde{K}_s) \equiv (0, 0, 0)$ is the unique solution of (6.24). Let

$$\begin{array}{l} \overline{P}_s = I_{[t,\tau]}(s)P_s + I_{(\tau,t+\delta]}(s)\widetilde{P}_s, \\ \overline{Q}_s = I_{[t,\tau]}(s)Q_s + I_{(\tau,t+\delta]}(s)\widetilde{Q}_s, \\ \overline{K}_s = I_{[t,\tau]}(s)K_s + I_{(\tau,t+\delta]}(s)\widetilde{K}_s,\ s \in [t, t+\delta]. \end{array}$$

It's easy to show that $(\overline{P}, \overline{Q}, \overline{K})$ is a solution of FBSDE (6.22). Therefore, from the uniqueness of solution of FBSDE (6.22) on $[t, t+\delta]$, where $0 \le \delta \le \delta_3$, we have $\overline{P}_t = P_t = 1 > 0$. Furthermore, from the definition of τ we have $\overline{P}_{t+\delta} \ge 0$, P-a.s., that is, $P_{t+\delta} \ge 0$, P-a.s. Therefore, we have $Y_t^1 \ge Y_t^2$, P-a.s. □

Proposition 6.5. *Let Φ be deterministic. We suppose the assumptions (C1), (C2), and (C4) hold true. Then, for every $p \geq 2$, there exists sufficiently small constant $\widetilde{\delta} > 0$, only depending on the Lipschitz constant K, and some constant $\tilde{C}_{p.K}$, only depending on p, the Lipschitz constant K and the linear growth constant L, such that for every $0 \leq \delta \leq \widetilde{\delta}$ and $\zeta \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}^n)$,*

$$\begin{array}{ll} \text{(i)} & E[\sup\limits_{t\leq s\leq t+\delta}|X_s^{t,\zeta}|^p + \sup\limits_{t\leq s\leq t+\delta}|Y_s^{t,\zeta}|^p + (\int_t^{t+\delta}|Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq \tilde{C}_{p,K}(1+|\zeta|^p),\ P\text{-a.s.}; \\ \text{(ii)} & E[\sup\limits_{t\leq s\leq t+\delta}|X_s^{t,\zeta}-\zeta|^p \mid \mathcal{F}_t] \leq \tilde{C}_{p,K}\delta^{\frac{p}{2}}(1+|\zeta|^p),\ P\text{-a.s.}; \\ \text{(iii)} & E[(\int_t^{t+\delta}|Z_s^{t,\zeta}|^2 ds)^{\frac{p}{2}} \mid \mathcal{F}_t] \leq \tilde{C}_{p,K}\delta^{\frac{p}{2}}(1+|\zeta|^p),\ P\text{-a.s.}, \end{array}$$

where $(X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})_{s\in[t,t+\delta]}$ is the solution of FBSDE (6.1) associated with $(b, \sigma, f, \zeta, \Phi)$ and with the time horizon $t+\delta$.

Proof. Due to Proposition 6.4 there exists a constant $\delta_0 > 0$, such that for every $0 \leq \delta \leq \delta_0$, (6.1) has a unique solution on $[t, t+\delta]$, i.e.,

$$Y_s^{t,\zeta} = \Phi(X_{t+\delta}^{t,\zeta}) + \int_s^{t+\delta} f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})dr - \int_s^{t+\delta} Z_r^{t,\zeta}dB_r,\ t \leq s \leq t+\delta. \tag{6.25}$$

We consider $\widetilde{Y}_s^{t,\zeta} = Y_s^{t,\zeta} - \Phi(\zeta)$, and for any $\beta \geq 0$ by applying Itô's formula to $e^{\beta s}|\widetilde{Y}_s^{t,\zeta}|^2$, we get

$$\begin{array}{rl} & E[e^{\beta s}|\widetilde{Y}_s^{t,\zeta}|^2|\mathcal{F}_s] + E[\int_s^{t+\delta}\{\beta e^{\beta r}|\widetilde{Y}_r^{t,\zeta}|^2 + e^{\beta r}|Z_r^{t,\zeta}|^2\}dr|\mathcal{F}_s] \\ = & E[e^{\beta(t+\delta)}|\widetilde{Y}_{t+\delta}^{t,\zeta}|^2|\mathcal{F}_s] + E[\int_s^{t+\delta} e^{\beta r}2\widetilde{Y}_r^{t,\zeta}f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta})dr|\mathcal{F}_s] \\ = & E[e^{\beta(t+\delta)}|\widetilde{Y}_{t+\delta}^{t,\zeta}|^2|\mathcal{F}_s] + E[\int_s^{t+\delta} e^{\beta r}2\widetilde{Y}_r^{t,\zeta}(f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}) - f(r,\zeta,\Phi(\zeta),0) + f(r,\zeta,\Phi(\zeta),0))dr|\mathcal{F}_s]. \end{array} \tag{6.26}$$

By taking β large enough and since $|f(r, \zeta, \Phi(\zeta), 0)| \leq C(1 + |\zeta|)$, we get by using BSDE standard methods

$$\begin{array}{l} |\widetilde{Y}_s^{t,\zeta}|^2 + E[\int_s^{t+\delta}\{|\widetilde{Y}_r^{t,\zeta}|^2 + |Z_r^{t,\zeta}|^2\}dr|\mathcal{F}_s] \\ \leq CE[\sup\limits_{s\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^2|\mathcal{F}_s] + C(t+\delta-s)(1+|\zeta|^2),\ \text{P-a.s.}, \end{array} \tag{6.27}$$

where C only depends on K and L. Therefore, from (6.25) and (6.27) and Buckholder-Davis-Gundy inequality,

$$E[\sup_{t\leq s\leq t+\delta}|\widetilde{Y}_s^{t,\zeta}|^2|\mathcal{F}_t] \leq CE[\sup_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^2|\mathcal{F}_t] + C\delta(1+|\zeta|^2),\ \text{P-a.s.} \tag{6.28}$$

On the other hand, from (6.27)

$$|\widetilde{Y}_s^{t,\zeta}|^2 \leq CE[\sup_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^2|\mathcal{F}_s] + C\delta(1+|\zeta|^2),\ \text{P-a.s.},\ t \leq s \leq t+\delta. \tag{6.29}$$

When $p > 2$, we define $\eta = \sup\limits_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta| \in L^2(\Omega, \mathcal{F}_{t+\delta}, P)$. Then $M_s := E[\eta|\mathcal{F}_s],\ s \in [t, t+\delta]$, is a martingale, and from Doob's martingale inequality, we have

$$\begin{array}{l} E[\sup\limits_{t\leq s\leq t+\delta}|M_s|^{\frac{p}{2}}|\mathcal{F}_t] \leq C_pE[|M_{t+\delta}|^{\frac{p}{2}}|\mathcal{F}_t] \leq C_pE[\eta^{\frac{p}{2}}|\mathcal{F}_t] \\ = C_pE[\sup\limits_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^p|\mathcal{F}_t],\ \text{P-a.s.} \end{array} \tag{6.30}$$

Therefore, from (6.29) and (6.30)

$$E[\sup_{t\leq s\leq t+\delta}|\widetilde{Y}_s^{t,\zeta}|^p|\mathcal{F}_t] \leq C_pE[\sup_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^p|\mathcal{F}_t] + C_p\delta^{\frac{p}{2}}(1+|\zeta|^p),\ \text{P-a.s.} \tag{6.31}$$

Now we consider $Y_s^{t,\zeta}-\Phi(\zeta)=\Phi(X_{t+\delta}^{t,\zeta})-\Phi(\zeta)+\int_s^{t+\delta}f(r,X_r^{t,\zeta},Y_r^{t,\zeta},Z_r^{t,\zeta})dr-\int_s^{t+\delta}Z_r^{t,\zeta}dB_r,\ t\leq s\leq t+\delta.$ From Burkholder-Davis-Gundy inequality and (6.31),

$$
\begin{aligned}
&E[(\textstyle\int_t^{t+\delta}|Z_s^{t,\zeta}|^2ds)^{\frac{p}{2}}\mid\mathcal{F}_t]\leq C_pE[\sup\limits_{t\leq s\leq t+\delta}|\textstyle\int_t^s Z_r^{t,\zeta}dB_r|^p\mid\mathcal{F}_t]\\
\leq\ & C_pE[\sup\limits_{t\leq s\leq t+\delta}|\widetilde{Y}_s^{t,\zeta}|^p+(\textstyle\int_t^{t+\delta}|f(s,X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})|ds)^p\mid\mathcal{F}_t]\\
=\ & C_pE[\sup\limits_{t\leq s\leq t+\delta}|\widetilde{Y}_s^{t,\zeta}|^p\mid\mathcal{F}_t]\\
&+C_pE[(\textstyle\int_t^{t+\delta}|f(s,X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})-f(s,\zeta,\Phi(\zeta),0)+f(s,\zeta,\Phi(\zeta),0)|ds)^p\mid\mathcal{F}_t]\\
\leq\ & (C_p+C_p\delta^p)E[\sup\limits_{t\leq s\leq t+\delta}|X_s^{t,\zeta}-\zeta|^p\mid\mathcal{F}_t]+C_p\delta^{\frac{p}{2}}(1+|\zeta|^p)+C_p\delta^{\frac{p}{2}}E[(\textstyle\int_t^{t+\delta}|Z_s^{t,\zeta}|^2ds)^{\frac{p}{2}}\mid\mathcal{F}_t].
\end{aligned}
$$

By choosing then $0<\delta_1\leq\delta_0$ such that $1-C_p\delta_1^{\frac{p}{2}}>0$, we get, for any $0\leq\delta\leq\delta_1$, P-a.s.,

$$
E[(\int_t^{t+\delta}|Z_s^{t,\zeta}|^2ds)^{\frac{p}{2}}\mid\mathcal{F}_t]\leq(C_p+C_p\delta^p)E[\sup_{t\leq s\leq T}|X_s^{t,\zeta}-\zeta|^p\mid\mathcal{F}_t]+C_p\delta^{\frac{p}{2}}(1+|\zeta|^p). \tag{6.32}
$$

Similarly, equation (6.1) and the estimates (6.31) and (6.32) yield

$$
\begin{aligned}
&E[\sup\limits_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^p\mid\mathcal{F}_t]\\
\leq\ & C_pE[(\textstyle\int_t^{t+\delta}b(r,X_r^{t,\zeta},Y_r^{t,\zeta},Z_r^{t,\zeta})dr)^p\mid\mathcal{F}_t]+C_pE[(\textstyle\int_t^{t+\delta}|\sigma(r,X_r^{t,\zeta},Y_r^{t,\zeta},Z_r^{t,\zeta})|^2dr)^{\frac{p}{2}}\mid\mathcal{F}_t]\\
\leq\ & C_p(1+|\zeta|^p)\delta^{\frac{p}{2}}+(C_p\delta^{\frac{p}{2}}+C_pL_\sigma^p)E[(\textstyle\int_t^{t+\delta}|Z_r^{t,\zeta}|^2dr)^{\frac{p}{2}}\mid\mathcal{F}_t]\\
&+C_p\delta^{\frac{p}{2}}E[\sup\limits_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^pdr\mid\mathcal{F}_t]\\
\leq\ & C_p(1+|\zeta|^p)\delta^{\frac{p}{2}}+(C_p\delta^{\frac{p}{2}}+C_pL_\sigma^p)E[\sup\limits_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^p\mid\mathcal{F}_t].
\end{aligned}
$$

Let $L_\sigma>0$ be sufficiently small such that $C_pL_\sigma^p<1$. Then there exists constant $0<\delta_2\leq\delta_1$ such that $1-(C_p\delta_2^{\frac{p}{2}}+C_pL_\sigma^p)>0$, and we obtain,

$$
E[\sup_{t\leq r\leq t+\delta}|X_r^{t,\zeta}-\zeta|^p\mid\mathcal{F}_t]\leq C_p(1+|\zeta|^p)\delta^{\frac{p}{2}},\ \text{P-a.s.},\ t\leq s\leq t+\delta. \tag{6.33}
$$

Finally, (6.31) and (6.32) allow to complete the proof. □

Similarly, we can prove the following proposition.

Proposition 6.6. *Suppose that $(b_i,\sigma_i,f_i,\Phi_i),\ i=1,2$ all satisfy the assumptions (C1), (C2) and (C4). There exists a constant $0<\delta_0$, only depending on the Lipschitz constant K, such that for every $0\leq\delta\leq\delta_0$, the same initial state $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$, $(X_s^i,Y_s^i,Z_s^i)_{s\in[t,t+\delta]}$ is the solution of FBSDE (6.1) associated with $(b_i,\sigma_i,f_i,\Phi_i)$ on the time interval $[t,t+\delta]$, respectively. Then we have: there exists a constant $\delta_1>0$, such that for every $0\leq\delta\leq\delta_1$,*

$$
\begin{aligned}
|Y_t^1-Y_t^2|^2\ \leq\ & CE[|\Phi_1(t+\delta,X_{t+\delta}^1)-\Phi_2(t+\delta,X_{t+\delta}^1)|^2\mid\mathcal{F}_t]\\
&+C\delta E[\textstyle\int_t^{t+\delta}|(b_1-b_2)(s,X_s^1,Y_s^1,Z_s^1)|^2ds\mid\mathcal{F}_t]\\
&+CE[\textstyle\int_t^{t+\delta}|(\sigma_1-\sigma_2)(s,X_s^1,Y_s^1,Z_s^1)|^2ds\mid\mathcal{F}_t]\\
&+C\delta E[\textstyle\int_t^{t+\delta}|(f_1-f_2)(s,X_s^1,Y_s^1,Z_s^1)|^2ds\mid\mathcal{F}_t],\ \text{P-a.s.}
\end{aligned}
$$

Remark 6.6. *When $(b_1,\sigma_1,f_1)=(b_2,\sigma_2,f_2)$ in Proposition 6.6, we have*

$$
|Y_t^1-Y_t^2|\leq C(E[|\Phi_1(t+\delta,X_{t+\delta}^1)-\Phi_2(t+\delta,X_{t+\delta}^1)|^2|\mathcal{F}_t])^{\frac{1}{2}},\ \text{P-a.s.}
$$

Corollary 6.1. *Under the assumptions (C1), (C2) and (C4), there exists a constant $0<\delta_0$, only depending on the Lipschitz constant K, such that for every $0\leq\delta\leq\delta_0$, the associated initial state $\zeta\in L^2(\Omega,\mathcal{F}_t,P;\mathbb{R}^n)$ and any $\varepsilon>0$, $(X_s^{t,\zeta},Y_s^{t,\zeta},Z_s^{t,\zeta})_{s\in[t,t+\delta]}$ is the solution of FBSDE (6.1) associated with (b,σ,f,ζ,Φ), and $(\overline{X}_s^{t,\zeta},\overline{Y}_s^{t,\zeta},\overline{Z}_s^{t,\zeta})_{s\in[t,t+\delta]}$ is the solution of FBSDE (6.1) associated with $(b,\sigma,f,\zeta,\Phi+\varepsilon)$ on the time interval $[t,t+\delta]$, respectively. Then we have*

$$
|Y_t^{t,\zeta}-\overline{Y}_t^{t,\zeta}|\leq C\varepsilon,\ \text{P-a.s.}
$$

Acknowledgments

The authors thank the referees for their very helpful comments. Juan Li acknowledges financial support by the NSF of P.R.China (No. 10701050, 11071144, 11222110), Shandong Province (Nos. Q2007A04, BS2011SF010, JQ201202), SRF for ROCS (SEM), 111 Project (No. B12023).

References

[1] F. ANTONELLI, *Backward-forward stochastic differential equations,* Ann. Appl. Probab., 3 (1993), pp. 777-793.

[2] R. BUCKDAHN and J. LI, *Stochastic differential games and viscosity solution of Hamilton-Jacobi-Bellman-Isaacs equation,* SIAM J. Control, Optim. 47 (2008), pp. 444-475.

[3] M.G. CRANDALL, H. ISHII and P.L. LIONS, *User's guide to viscosity solutions of second order partial differential equations,* Bull. Amer. Math. Soc. 27 (1992), pp. 1-67.

[4] J. CVITANIĆ and J. MA, *Hedging options for a large investor and forward-backward SDEs,* Ann. Appl. Probab. 6(2) (1996), pp. 370-398.

[5] F. DELARUE, *On the existence and uniqueness of solutions to FBSDEs in a non-degenerate Case*, Stoch. Process. Appl. 99 (2002), pp. 209-286.

[6] N. El KAROUI, S.G. PENG, and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance 7 (1997), pp. 1-71.

[7] Y. HU and S.G. PENG, *Solutions of forward-backward stochastic differential equations*, Probab. Theory and Rel. Fields 103 (1995), pp. 273-283.

[8] J. MA, P. PROTTER, and J.M. YONG, *Solving forward-backward stochastic differential equations explicitly-a four step scheme*, Probab. Theory Relat. Fields 98 (1994), pp. 339-359.

[9] J. MA, Z. WU, J. ZHANG, and D. ZHANG, *On wellposedness of forward-backward SDEs-a unified approach*, arxiv.org/abs/1110.4658

[10] J. MA and J.M. YONG, *Forward-backward stochastic differential equations and their applications*, Springer, Berlin, 1999.

[11] E. PARDOUX and S.G. PENG, *Adapted solution of a backward stochastic differential equation,* Systems Control Lett. 14(1-2) (1990), pp. 55-61.

[12] E. PARDOUX and S.J. TANG, *Forward-backward stochastic differential equations and quasilinear parabolic PDEs,* Prob. Theory Relat. Fields 114 (1999), pp. 123-150.

[13] S.G. PENG, *A general stochastic maximum principle for optimal control problems,* SIAM J. Control Optim. 28 (1990), pp. 966-979.

[14] S.G. PENG, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equation,* Stochastic Reports 37 (1991), pp. 61-74.

[15] S.G. PENG, *A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation,* Stochastics and Stochastics Reports. 38 (1992), pp. 119-134.

[16] S.G. PENG, (1997) *Backward stochastic differential equations-stochastic optimization theory and viscosity solutions of HJB equations, in Topics on stochastic analysis,* J.A. YAN, S.G. PENG, S.Z. FANG, and L.M. WU, Ch.2, Science Press. Beijing, 1997, pp. 85-138 (in Chinese).

[17] S.G. PENG and Z. WU, *Fully coupled forward-backward stochastic differential equations and applications to optimal control*, SIAM Control Optim. 37(3) (1999), pp. 825-843.

[18] Z. WU, *The comparison theorem of FBSDE,* Statist. Probab. Lett. 44 (1999), pp. 1-6.

[19] Z. WU and Z.Y. YU, *Fully coupled forward-backward stochastic differential equations and associated partial differential equation systems*, Annals of Mathematics 25A:4 (2004), pp. 457-468 (in Chinese).

[20] Z. WU and Z.Y. YU, (2010) *Probabilistic Interpretation for systems of Parabolic Partial Differential Equations Combined with Algebra Equations*, preprint.

[21] J.M. YONG, *Finding adapted solutions of forward-backward stochastic differential equations: method of continuation,* Probab. Theory Relat. Fields. 107 (1997), pp. 537-572.

[22] J.M. YONG, *Forward-backward stochastic differential equations with mixed initial-terminal conditions* Transactions of the American Mathematical Society. 362(2) (2010), pp. 1047-1096.

[23] J.F. ZHANG, *The wellposedness of FBSDEs*, Discrete Contin. Dyn. Syst., Ser. B 6 (2006), pp. 927-940.